



Experimental Study of the Convergence of the Chebyshev Method

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Abstract. This paper presents an experimental investigation of the convergence properties of the Chebyshev spectral method for solving ordinary differential equations. The method is based on approximating the target solution by an expansion in Chebyshev polynomials and applying the collocation approach at Chebyshev–Lobatto nodes. The numerical experiments confirm the theoretical result concerning the exponential convergence of spectral methods for analytic functions. In particular, it is demonstrated that the logarithm of the approximation error decreases linearly as the degree of the approximating polynomial increases. Furthermore, the influence of computational precision on the observed convergence rate is examined.

Keywords: Chebyshev spectral method, Richardson method, linear ordinary differential equation.

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1. Introduction

Numerical methods for solving ordinary differential equations play a crucial role in many areas of science and engineering. Mathematical models arising in physics, mechanics, biology, and economics often lead to differential equations whose analytical solutions are either impossible or extremely difficult to obtain. In such cases, various numerical methods are employed to construct approximate solutions.

Among the wide variety of numerical approaches, spectral methods occupy a special place due to their representation of the solution as an expansion in terms of a set of global basis functions. In contrast to classical finite-difference or finite-element methods, spectral methods utilize global approximations, which makes it possible to achieve significantly higher accuracy with a relatively small number of degrees of freedom.

One of the most widely used spectral approaches is based on Chebyshev polynomials. Owing to the special properties of these polynomials, the Chebyshev method possesses high numerical stability and demonstrates exponential convergence when approximating sufficiently smooth functions [1].

The aim of this work is an experimental investigation of the convergence properties of the Chebyshev method for solving ordinary differential equations. A particular test problem is considered, for which a numerical approximation of the solution is constructed using a collocation approach. The dependence of the approximation error on the degree of the approximating polynomial is then analyzed, and the influence of computational precision on the observed convergence rate of the method is investigated.

2. Problem Formulation

Consider the first-order ordinary differential equation

$$\frac{dy}{dx} = f(x, y),$$

supplemented with the initial condition

$$y(x_0) = y_0.$$

Such problems belong to the class of initial value problems (Cauchy problems) and frequently arise in various applied models.

In this work, the following equation is considered as a test example:

$$\frac{dy}{dx} = xy - x^2,$$

with the initial condition

$$y(0) = 1.$$

This equation was chosen for several reasons. First, it is sufficiently simple for numerical experimentation. Second, its solution is a smooth function, making the

problem well suited for the investigation of spectral methods, which exhibit their highest efficiency for smooth or analytic solutions.

The main objective of this study is to solve the given equation numerically using the Chebyshev method and to experimentally investigate the convergence rate of the approximation as the degree of the approximating polynomial increases. To this end, the dependence of the logarithm of the approximation error on the number of collocation nodes N is analyzed. If the method exhibits exponential convergence, this dependence should be close to linear.

3. Chebyshev Method

The Chebyshev method belongs to the class of spectral methods and is based on representing the desired solution as an expansion in terms of Chebyshev polynomials of the first kind [2, 3].

The Chebyshev polynomials $T_k(x)$ form an orthogonal system of functions on the interval $[-1, 1]$ and possess a number of important properties that make them particularly convenient for numerical methods. In particular, they minimize oscillatory effects (the Runge phenomenon) and provide good numerical stability of the approximation [4, 5, 6, 7].

The desired solution $y(x)$ is approximated by a polynomial of the form

$$p(x) = \sum_{k=0}^N c_k T_k(x),$$

where c_k are unknown coefficients and N is the degree of the approximating polynomial.

The coefficients are determined using the collocation method. The main idea of the method is that the approximating polynomial is required to satisfy the differential equation not on the entire interval, but only at a finite number of specially selected points called collocation points.

As such points, the Chebyshev–Gauss–Lobatto nodes are commonly used [2, 3]:

$$x_j = \cos\left(\frac{j\pi}{N-1}\right), \quad j = 0, 1, \dots, N-1.$$

The choice of these points is motivated by the fact that they provide optimal interpolation properties and minimize numerical errors.

Substituting the approximation $p(x)$ into the original differential equation and requiring the equation to hold at the collocation points yields a system of algebraic equations with respect to the coefficients c_k .

An additional condition [4], corresponding to the initial condition of the problem, is added to this system:

$$\sum_{k=0}^N c_k T_k(x_0) = y_0.$$

Thus, a system of $N + 1$ equations with respect to $N + 1$ unknown coefficients is obtained. Solving this system makes it possible to determine the expansion coefficients and construct an approximate solution of the differential equation.

4. Theoretical Foundations of Exponential Convergence of Chebyshev Approximation

One of the principal advantages of spectral methods based on Chebyshev polynomials is their exponential convergence rate when approximating sufficiently smooth functions. In contrast to classical algebraic approximation methods, where the error decreases according to a power law of the form N^{-p} , spectral methods provide a significantly faster reduction of the approximation error as the number of basis functions increases.

Exponential convergence means that the approximation error decreases according to the relation

$$\|f - f_N\| \leq Ce^{-\alpha N},$$

where C and α are positive constants independent of N . Such behavior implies that high approximation accuracy can be achieved even for relatively small values of N .

Fundamental results describing this property were established within approximation theory. A detailed exposition of these results can be found, for example, in [1, ch. 8]. In particular, Theorems 8.1–8.3 establish a connection between the analyticity of a function and the decay rate of the coefficients in its Chebyshev expansion.

In contrast, for problems involving discontinuities in the coefficients or in the right-hand side of a boundary value problem, the convergence of the Chebyshev spectral method becomes algebraic, and its order is determined by the smoothness of the solution [1, Ch. 7].

The main idea is as follows. If a function $f(x)$ is analytic in a region of the complex plane containing the interval $[-1, 1]$, then the coefficients of its Chebyshev expansion decay exponentially. Since the truncation error is determined by the sum of the “tail” coefficients, this leads to an exponential decrease in the approximation error.

It is precisely this property that makes spectral methods based on Chebyshev polynomials particularly effective for solving problems in mathematical physics and differential equations, where the solutions are typically sufficiently smooth or analytic functions.

5. Rigorous Proof of Exponential Convergence

Consider a real-valued function $f(x)$ defined on the interval $[-1, 1]$. Assume that this function is analytic on the interval and admits an analytic continuation into

the interior of the open Bernstein ellipse E_ρ with parameter $\rho > 1$. Suppose that within this ellipse the inequality

$$|f(x)| \leq M$$

holds for some constant M .

Due to analyticity, the function $f(x)$ can be represented as an infinite Chebyshev series

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

where $T_k(x)$ are Chebyshev polynomials of the first kind and a_k are the expansion coefficients.

The approximation of order n is defined as the partial sum

$$f_n(x) = \sum_{k=0}^n a_k T_k(x).$$

For analytic functions, the Chebyshev coefficients decay exponentially. In particular, the following estimates hold:

$$|a_0| \leq M, \quad |a_k| \leq 2M\rho^{-k}, \quad k \geq 1.$$

Observe that the expression ρ^{-k} can be written in exponential form:

$$\rho^{-k} = e^{-k \ln(\rho)}.$$

The approximation error is given by

$$E_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x).$$

An important property of Chebyshev polynomials is

$$|T_k(x)| \leq 1, \quad x \in [-1, 1].$$

Using this property, we obtain the estimate

$$|E_n(x)| \leq \sum_{k=n+1}^{\infty} |a_k|.$$

Therefore, for the uniform norm we have

$$\|f - f_n\|_{\infty} = \max_{x \in [-1, 1]} |E_n(x)| \leq \sum_{k=n+1}^{\infty} |a_k|.$$

Substituting the estimate for the coefficients yields

$$\|f - f_n\|_\infty \leq \sum_{k=n+1}^{\infty} 2M\rho^{-k} = 2M \sum_{k=n+1}^{\infty} \left(\frac{1}{\rho}\right)^k.$$

The right-hand side is a geometric series with ratio $1/\rho < 1$. Its sum is

$$\sum_{k=n+1}^{\infty} \left(\frac{1}{\rho}\right)^k = \frac{\rho^{-(n+1)}}{1 - \rho^{-1}} = \frac{\rho^{-n}}{\rho - 1}.$$

Consequently,

$$\|f - f_n\|_\infty \leq \frac{2M\rho^{-n}}{\rho - 1}.$$

Passing to exponential form,

$$\rho^{-n} = e^{-n \ln(\rho)},$$

we obtain the final estimate

$$\|f - f_n\|_\infty \leq \frac{2M}{\rho - 1} e^{-n \ln(\rho)}.$$

Introducing the notation

$$C = \frac{2M}{\rho - 1}, \quad \alpha = \ln(\rho) > 0,$$

we arrive at the final expression

$$\|f - f_n\|_\infty \leq Ce^{-\alpha n}.$$

This inequality expresses the exponential convergence of the Chebyshev approximation.

6. Numerical Experiments and Convergence Analysis of the Chebyshev Method

The computations were performed using the Sage computer algebra system [8]. This section presents the results of numerical experiments aimed at investigating the convergence of the Chebyshev method for solving ordinary differential equations. The primary objective of the experiments is to provide an experimental confirmation of the theoretical result concerning the exponential convergence of spectral methods, as well as to determine the constants C and α characterizing the convergence rate.

To estimate the accuracy, the approximation error is evaluated using Richardson extrapolation [9, 10, 11, 12, 13]. For various values of the approximating polynomial degree N , the error estimate is computed at several fixed points of the domain:

$$x \in \{1, 0.5, -0.5\}.$$

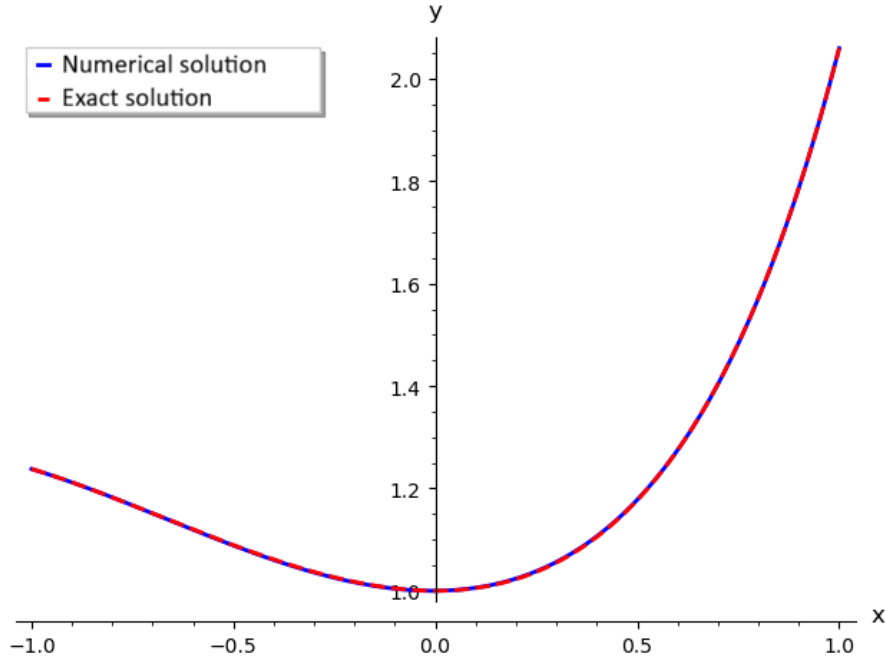


Figure 1: Graph of the exact solution of the problem $y'(x) = xy - x^2$, $y(0) = 1$

This approach makes it possible to investigate the convergence of the method not only at a single point, but also in different parts of the interval, including interior points and neighborhoods of the boundary.

For each selected point, the dependence of $\log(\text{error})$ on N is constructed.

If the error decreases according to the exponential law

$$\|f - f_N\| \approx Ce^{-\alpha N},$$

then, after taking the logarithm, one obtains the linear relation

$$\log(\|f - f_N\|) \approx -\alpha N + \log C.$$

Thus, the linear behavior of the graph of $\log(\text{error})$ versus N serves as an experimental confirmation of the exponential convergence of the method.

Example 1. Consider the problem

$$\frac{dy}{dx} = xy - x^2, \quad y(0) = 1.$$

The exact solution of this equation can be obtained analytically. Figure 1 shows the graph of the exact solution on the interval $[-1, 1]$. This graph is used as a reference for evaluating the accuracy of the numerical solution obtained by the Chebyshev method. To solve this equation, the collocation method at Chebyshev–Lobatto nodes is employed. The approximate solution is taken as a Chebyshev polynomial of

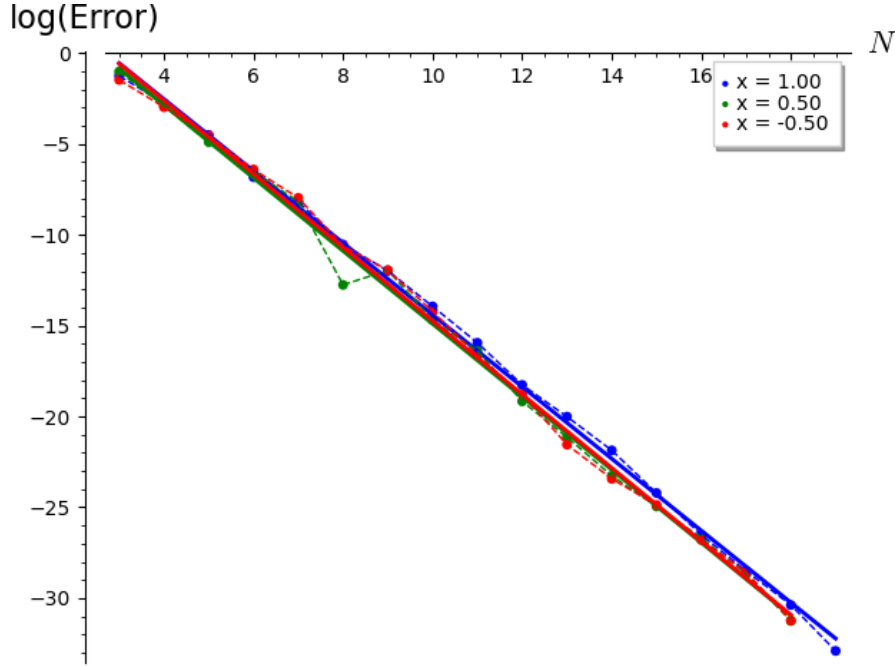


Figure 2: Dependence of $\log(\text{error})$ on the polynomial degree N for different values of x using `RealField(120)` precision

degree N , whose coefficients are determined from the system of algebraic equations obtained from the collocation conditions and the initial condition. Figure 2 shows the dependence of $\log(\text{error})$ on the polynomial degree N , computed at the points $x = 1$, $x = 0.5$, and $x = -0.5$. Each curve corresponds to a fixed value of x .

As can be seen from Figure 2, an almost linear dependence of $\log(\text{error})$ on N is observed for all selected points. This indicates that the approximation error decreases according to the exponential law

$$\|f - f_N\| \approx Ce^{-\alpha N}.$$

It should be noted that the convergence rate depends only slightly on the selected point x ; however, in all cases the exponential character of the error decay is preserved. A comparison between the numerical and exact solutions demonstrates that the approximation accurately reproduces the behavior of the function over the entire interval under consideration. Thus, the obtained results agree with the theoretical estimate in the uniform norm and confirm that the Chebyshev method provides exponential convergence on the entire interval $[-1, 1]$. The straight line in the graph is obtained by fitting the experimental data using the least-squares method:

$$\log(\text{error}) \approx aN + b.$$

From this approximation, the parameters of exponential convergence are determined

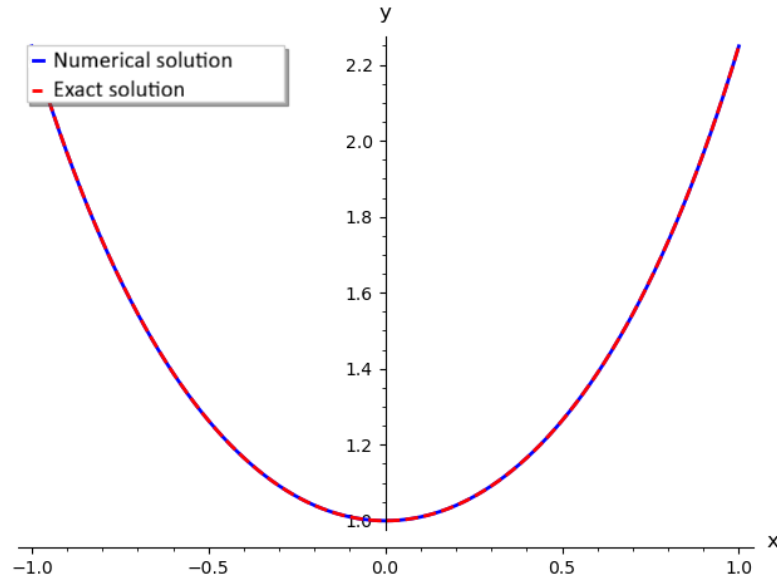


Figure 3: Graph of the exact solution of the linear equation

as

$$\alpha = -a, \quad C = e^b.$$

For this equation, we obtain

$$\alpha \approx 1.12, \quad C \approx 5.8 \times 10^{-6}.$$

Example 2. Consider the first-order linear equation

$$y'(x) - x y(x) = \sin(x), \quad y(0) = 1. \quad (1)$$

To illustrate the behavior of the solution, Figure 3 presents the graph of the exact solution of this equation. The solution is an analytic function on the interval $[-1, 1]$, which guarantees the exponential convergence of the Chebyshev method. For the numerical solution, the collocation method at Chebyshev–Lobatto nodes is employed. Figure 4 shows the dependence of $\log(\text{error})$ on the polynomial degree N , computed at the points $x = 1$, $x = 0.5$, and $x = -0.5$. The experimental points are located approximately along a straight line, confirming the exponential character of the error decay. The least-squares approximation gives

$$\log(\text{error}) \approx aN + b,$$

hence,

$$\alpha = -a, \quad C = e^b.$$

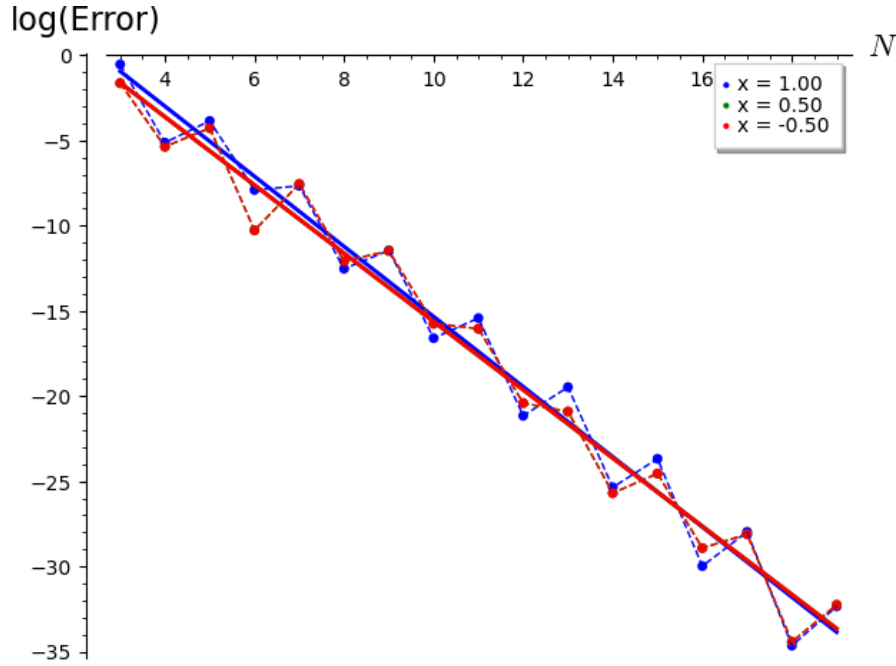


Figure 4: Dependence of $\log(\text{error})$ on N for the linear equation

For this equation, we obtain

$$\alpha \approx 0.78, \quad C \approx 4.2.$$

Even for relatively small values of N , high accuracy is achieved, demonstrating the efficiency of the Chebyshev method.

6.1 Investigation of Convergence for Nonsmooth Coefficients

When the coefficients are not smooth (for example, discontinuous or insufficiently differentiable), the exponential convergence of spectral methods is replaced by algebraic convergence. In this case, the approximation error decreases as a power function of N :

$$\|f - f_N\| \approx CN^{-k},$$

where k depends on the smoothness of the solution. Taking the logarithm yields a linear relationship on a log-log scale:

$$\log(\|f - f_N\|) \approx -k \log N + \log C.$$

Thus, the graph of $\log(\text{error})$ versus $\log N$ becomes a straight line, which is a characteristic indicator of algebraic convergence.

Consider the equation with a nonsmooth coefficient:

$$\frac{dy}{dx} = |x - 0.5| y + x^2, \quad y(0) = 0.$$

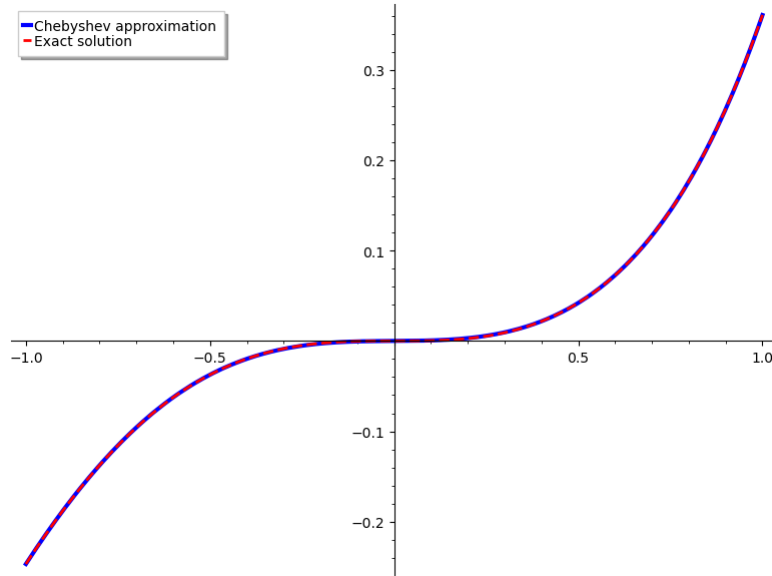


Figure 5: Comparison of the numerical solution obtained by the Chebyshev method with the reference solution

Unlike the previous examples, the coefficient $|x - 0.5|$ is not an analytic function at the point $x = 0.5$. Although the solution itself remains continuous, the smoothness of the problem deteriorates, which makes it possible to investigate the influence of nonsmoothness on the convergence rate of the Chebyshev method.

For the numerical solution, the same collocation method at Chebyshev–Lobatto nodes is employed. Since the equation is nonlinear with respect to the expansion coefficients, the resulting system of collocation equations is solved using Newton’s method.

Figure 5 shows a comparison between the numerical solution obtained by the Chebyshev method and a reference solution computed using the Runge–Kutta method.

As can be seen from Figure 5, the Chebyshev method reproduces the solution accurately over the entire interval $[-1, 1]$. To investigate the convergence rate, consider the dependence of $\log(\text{error})$ on the polynomial degree N , see Figure 6. In contrast to the previous examples, the graph is no longer strictly linear. The rate of error decay decreases noticeably as N increases. This behavior is explained by the fact that the function $|x - 0.5|$ is not analytic. According to the theory of spectral methods, exponential convergence is achieved only for analytic functions. In the presence of points of nonsmoothness, the convergence rate becomes algebraic. Nevertheless, the Chebyshev method continues to provide high solution accuracy, and the error still decreases as the number of basis functions increases. Thus, this experiment confirms a fundamental theoretical fact: the analyticity of the coefficients and the solution is a key condition for the exponential convergence of spectral methods.

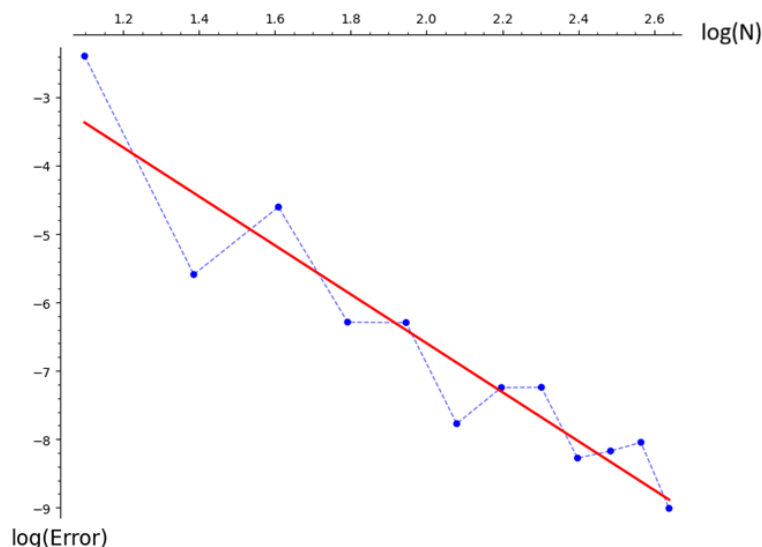


Figure 6: Dependence of $\log(\text{error})$ on N for the equation with a nonsmooth coefficient

7 Influence of Computational Precision: The Rounding Error Effect

We now investigate the influence of limited computational precision. For this purpose, the same experiment is repeated using numbers with lower precision: `RealField(20)` instead of `RealField(120)`.

The result is shown in Figure 7. As can be seen from the figure, in the initial part of the graph (for small values of N), the same behavior as in the previous experiment is observed: the points lie approximately along a straight line. This indicates that, for small values of N , the approximation error remains the dominant source of inaccuracy.

However, as N increases further, the behavior of the graph changes significantly. The experimental points begin to deviate from the straight line, and the dependence reaches a so-called “plateau”. This phenomenon is associated with the fact that the rounding error starts to dominate over the approximation error.

In other words, as the polynomial degree increases, the approximation error continues to decrease, but the computations are performed with limited precision. When the approximation error becomes comparable to, or smaller than, the rounding error, further increasing N no longer improves the result.

In some cases, the total error may even begin to increase due to the accumulation of numerical errors arising in the solution of the algebraic system for the expansion coefficients.

Thus, this experiment clearly demonstrates an important practical effect: when using spectral methods with very high convergence rates, it is necessary to employ sufficiently high numerical precision. Otherwise, the exponential convergence of the

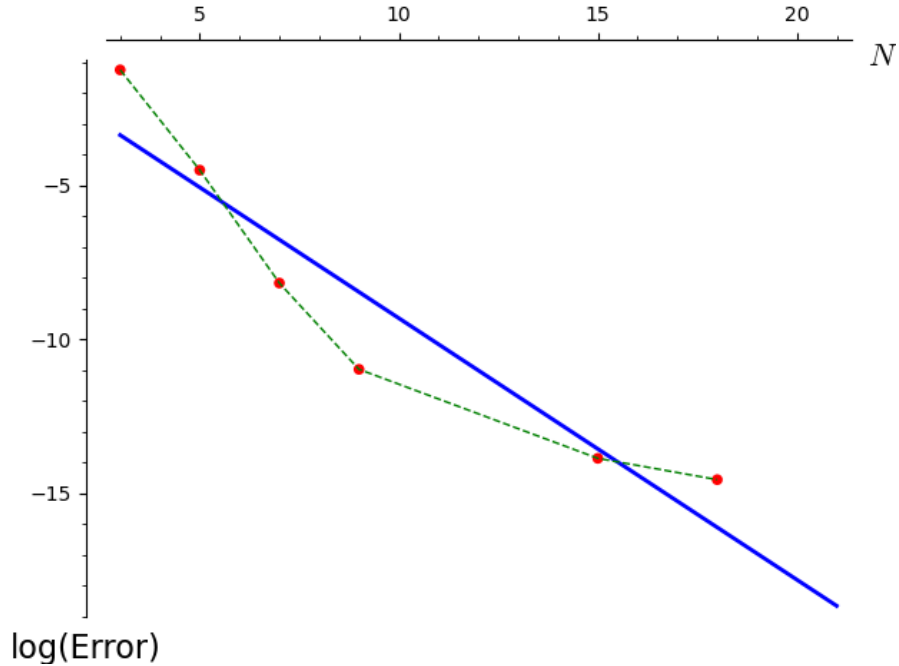


Figure 7: Dependence of $\log(\text{error})$ on N for computations performed with `RealField(20)` precision

method may be limited by the influence of rounding errors.

8. Conclusion

The conducted experimental study of the convergence of the Chebyshev method for solving ordinary differential equations clearly demonstrates its key advantage—exponential convergence. The graphs of the dependence of the logarithm of the error on the degree of the approximating polynomial N exhibit straight-line behavior, which provides direct evidence of exponential decay of the error as N increases.

This implies that a relatively small number of basis functions (Chebyshev polynomials) is sufficient to achieve high accuracy.

Overall, the Chebyshev method is a powerful and efficient tool for the numerical solution of ordinary differential equations, particularly for problems requiring high accuracy and involving sufficiently smooth solutions.

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