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Darboux Ideals of Hamiltonian Systems

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Abstract. For Hamiltonian systems, a generalization of the concept of Darboux polynomial is proposed. By analogy with the differential ideal, the concept of the Darboux ideal was introduced. The basic properties of such ideals are indicated. The principal Darboux ideal is generated by a Darboux polynomial. For the Darboux ideal of the form $\langle f, g\rangle$, an analog of Jacobi–Poisson theorem is proved. If $J = \langle f, g\rangle$ is a Darboux ideal, then the ideal $\langle f, g, [f, g]\rangle$ is also a Darboux ideal of this ring. Here $[.,.]$ are designated Poisson brackets.

Keywords: Darboux polynomials, ideal, Poisson brackets.

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1. Introduction

Darboux polynomials, also known as partial integrals of dynamical systems, were introduced in the 19th century, when it was hoped that rational integrals of motion could be computed from known Darboux polynomials [1, 2]. They occupied an important place in Lagutinskii's theory [3, 4, 5, 6]. Currently, Darboux polynomials of Hamiltonian systems are actively studied by the research group at Grodno University [7, 8, 9].

Our interest in Darboux polynomials is related to the development of geometric integrators [10]. The development of the integrators preserving the symplectic structure of Hamiltonian systems began in the 1990s in the papers of Yu. B. Suris [11] and Cooper [12]. Unfortunately, hopes that preserving one algebraic structure would imply that a difference scheme inherits all other algebraic properties of the continuous dynamical model were not fulfilled [10]. The preservation of Darboux polynomials on difference schemes has remained in the shadow of the study of the inheritance of motion integrals.

In this paper, we construct a multidimensional analogue of the theory of Darboux integrals — the theory of Darboux ideals of Hamiltonian systems. We believe that this concept has not been sufficiently formalized previously, and the question of the inheritance of Darboux ideals by difference schemes has not been raised before.

2. Darboux Ideals

Let a Hamiltonian system be given by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n, \tag{1}$$

and let the Hamiltonian $H(p, q, t)$ be a polynomial in $q_1, \dots, p_n$, whose coefficients are meromorphic functions of t. Denote the ring of such polynomials by K, and their coefficient field by k [13].

Introduce the differentiation, as usual

$$D = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

on the ring K. We believe that the concept of a Darboux polynomial can be generalized as follows.

Definition 1. *An ideal J of the ring K will be called a Darboux ideal of K with differentiation D if $f \in J$ implies $Df \in J$.*

If Darboux ideal J is principal, i.e. $J = \langle g \rangle$, then

$$Dg = h \cdot g$$

for some $h \in K$ and therefore g is a Darboux polynomial [1, §2.21], also called a second integral [2, §2.5] or partial integral [7]. This circumstance motivates our generalization (Def. 1).

The property — $DJ \subset J$ — is the same that is used in the definition of the differential ideal, introduced firstly in [14]. In our opinion, Darboux ideals and differential ideals should not be called the same, since they use different rings. The definition (1) of the Darboux ideal uses the standard polynomial ring $k[q_1, \ldots, p_n]$ rather than the differential ring.

Every ideal J of the ring K is generated by a finite number of elements, say,

$$J = \langle g_1, \ldots, g_r \rangle,$$

then the equations

$$g_1 = 0, \ldots, g_r = 0$$

define an algebraic set in the affine space k^{2n}. We denote this set by $Z(J)$. Recall that, strictly speaking, the set $Z(J)$ can be called a variety only when J is a prime ideal [15].

Definition 2. *A set Z of points in the affine space k^{2n} will be called invariant if every particular solution $p(t), q(t)$ of the Hamiltonian system (1) that passes through a point $(p_0, q_0) \in Z|_{t=t_0}$ at some $t = t_0$ lies entirely on this set: $(p(t), q(t)) \in Z$.*

Theorem 1. *If J is a Darboux ideal of the ring K with differentiation D, then the set $Z(J)$ is integral for the Hamiltonian system (1).*

Proof. Let $f \in J$ and $p(t), q(t)$ be an arbitrary solution of the Hamiltonian system (1), that passes through a point $(p_0, q_0) \in Z(J)|_{t=t_0}$ at $t = t_0$. Consider

$$F(t) = f(p(t), q(t), t).$$

By Taylor's formula

$$F(t) = F(t_0) + F'(t_0)(t - t_0) + \ldots$$

At $t = t_0$, the solution passes through a point of $Z(J)|_{t=t_0}$, so $F(t_0) = 0$. Furthermore,

$$F'(t) = Df|_{t=t_0, p=p_0, q=q_0}$$

Therefore, from $Df \in J$, it follows that $F'(t_0) = 0$. Continuing in this manner, we see that F is identically zero. Hence, every polynomial from J vanishes on the considered particular solution (1). □

As usual [15], to each set Z we assign its ideal $I(Z)$, i.e., the set of polynomials from K that vanish on Z.

Theorem 2. *If Z is an invariant algebraic set for the Hamiltonian system* (1)*, then $I(Z)$ is a Darboux ideal of the ring K with differentiation D.*

Proof. Let $f \in I(Z)$ and $p(t), q(t)$ be an arbitrary solution of the Hamiltonian system (1) that passes through a point $(p_0, q_0) \in Z(J)|_{t=t_0}$ at $t = t_0$. By assumption,

$$F(t) = f(p(t), q(t), t)$$

is identically zero. Consequently, its derivative is also zero

$$F'(t) = Df(p(t), q(t), t)$$

everywhere, in particular at $t = t_0, p = p_0, q = q_0$. Since this point was chosen arbitrarily on Z, the polynomial Df vanishes on all of Z, i.e., $Df \in I(Z)$. Hence, $I(Z)$ is a Darboux ideal. □

Although k is not algebraically closed, Hilbert's Nullstellensatz holds at any fixed time moment t and therefore $I(Z(J)) = \sqrt{J}$, see [15]. If J is a Darboux ideal, then $Z(J)$ is an invariant set by Theorem 1, and thus $\sqrt{J} = I(Z(J))$ is a Darboux ideal by Theorem 2.

3. Interpretation of Poisson Brackets

In [8, 9] the question was formulated regarding the meaning of Poisson brackets of partial integrals, often also called Darboux integrals. Sufficient conditions were found under which the Poisson bracket is also a partial integral [9, Prop. 1]. What is the meaning of the Poisson bracket computed for equations defining integral manifolds of arbitrary dimension?

Let us turn to the case of a single Darboux ideal generated by two elements, i.e., $J = \langle f, g \rangle$. Introduce the Poisson bracket

$$[f, g] = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

on the ring K.

Theorem 3. *If $J = \langle f, g \rangle$ is a Darboux ideal of the ring K with differentiation D, then the ideal*

$$\langle f, g, [f, g] \rangle$$

is also a Darboux ideal of this ring.

Proof. In [9, p. 15] the identity

$$D[f, g] = [f, Dg] + [Df, g] \tag{2}$$

was proved.

If J is a Darboux ideal, then $Dg, Df \in J$, i.e.,

$$Df = h_{11}f + h_{12}g, \quad Dg = h_{21}f + h_{22}g,$$

and therefore

$$D[f,g] = (h_{12} + h_{21})[f,g] + ([f,h_{21}] + [h_{11},g])f + ([f,h_{22}] + [h_{12}])g.$$

Then, for any $h \in K$, we have

$$D(h[f,g]) = (Dh + h(h_{12} + h_{21}))[f,g] + h([f,h_{21}] + [h_{11},g])f + h([f,h_{22}] + [h_{12}])g.$$

This means that the derivative of any element of the ideal

$$\langle f, g, [f,g] \rangle$$

belongs to this ideal. □

The formula $J = \langle f, g \rangle$ means that in the space (p, q, t) of dimension $2n+1$, there is an invariant set

$$f(p,q,t) = 0, \quad g(p,q,t) = 0$$

of dimension $2n-1$. The proved theorem allows reducing the dimension of the invariant set by 1 and considering

$$f = 0, \quad g = 0, \quad [f,g] = 0$$

as an invariant manifold. It is easy to see that this almost repeats the formulation of the classical Poisson theorem for integrals of motion.

4. Discussion

The concept of a Darboux ideal in the formulation of the Def. 1 belongs to the theory of commutative rings, while Theorem 1 connects this construction with the invariant manifolds of classical mechanics, thereby indicating another direction for applying computer algebra tools to the integration of dynamical systems.

The theorem on Poisson brackets is one of the fundamental theorems of classical mechanics. The very possibility of its generalization to Darboux ideals (Theorem 3) indicates that these ideals possess rich nontrivial properties that remain to be studied. The meaning of the original Poisson theorem is that, given two known integrals of motion, a third can be computed as their Poisson bracket. Our Theorem 3 can also be given such a form: from known equations $f = 0$ and $g = 0$, defining an invariant manifold of codimension 2, one can compute a third equation as the Poisson bracket, which together with $f = 0$ and $g = 0$ defines an invariant manifold of codimension 3. Proceeding in this manner, one can either reach a one-dimensional manifold (i.e., reduce the Hamiltonian system to quadratures) or, as usually happens with integrals, obtain a bracket that belongs to the original ideal. This should not discourage us, since in classical mechanics one usually cannot find all remaining integrals by successively computing Poisson brackets from two known integrals of motion.

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