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Using helical flows to test difference schemes for Navier-Stokes equations

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Abstract. Currently, a number of difference schemes are known that approximate the Navier-Stokes equations. Difference schemes are typically tested on two-dimensional problems, such as calculating the Karman street. However, three-dimensional flows present the greatest challenge. In the paper by V. P. Kovalev et al. (2017), two particular solutions to the Navier-Stokes equations were analytically described: the ABC solution and the Gromeka-Beltrami solutions. Both of these solutions describe helical flows, that is, flows in which the velocity curl curl $\vec{v}$ is proportional to the velocity $\vec{v}$. These flows provide a natural test case for comparing the performance of different difference schemes. Here we present the results of testing the schemes from the paper by V. P. Gerdt et al (2020) on these helical flows.

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1 Introduction

The need to study complex three-dimensional incompressible fluid flows arises in many applied problems. These include the problem of the structure of swirling flow in a pipe and the coupling of a dam's eddy flow with the downstream surface, which have been extensively studied both theoretically and experimentally at RUDN University [1, 2]. However, the numerical analysis of such flows still presents significant difficulties and necessitates the search for new numerical methods for studying mathematical models of swirling flows.

Consider fluid flow in a certain domain G. We introduce a fixed Cartesian coordinate system xyz. We denote the instantaneous fluid velocity at point (x, y, z) at time t as $\vec{v}(x, y, z, t)$. The dynamics of the velocity field can be described by the Navier-Stokes equations:

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v} \cdot \nabla)\vec{v} + \nu \Delta \vec{v} - \frac{1}{\rho} \nabla p, \tag{1}$$

where ν is the kinematic viscosity coefficient, ρ is the density, and p is the pressure. For water flow, ν and ρ can be considered known constants, while the pressure p is, in addition to the three velocity components, another unknown function. This results in three differential equations for four unknown functions. For an incompressible fluid, these equations are supplemented by the continuity equation:

$$\operatorname{div} \vec{v} = 0.$$

From a mathematical point of view, the Navier-Stokes equations are extremely complex; even the existence and smoothness of a solution to the Navier-Stokes equations in $\mathbb{R}^3$ has not yet been proven. Boundary value problems for partial differential equations are rarely solved analytically, so numerical, primarily grid-based, methods for solving differential equations are commonly used. Models based on the numerical solution of the Navier-Stokes equations are known in the English-language literature as Direct Numerical Simulation (DNS). These models are used in academic research at low Reynolds numbers, especially for gaseous flows. An example is the very popular two-dimensional problem of Karman street formation [3, §9.6].

The use of grid methods implies that, at a scale of the order of the grid step, the solution can be well approximated by linear functions. In turbulent problems, this assumption is not met; the Navier-Stokes equations apparently describe fine flow structure well and, for this reason, are very difficult to solve numerically using the finite difference method.

In the last century, a whole family of semi-empirical models was proposed that have proven themselves in engineering practice. Wilcox's description of these models has now become the standard; see also [4]. A classification of the various models is provided in the review [5]. The main problem with semi-empirical models is the need to select constants empirically. This problem is currently being addressed by refining the parameters during calculations using the method described in [6].

One possible way to preserve the scientific validity of models based on the Navier-Stokes equations and accurately describe turbulent structures is to develop mimetic difference schemes for the Navier-Stokes equationsthat is, difference schemes that accurately preserve certain structures associated with the Navier-Stokes equations. A difference scheme approximating the Navier-Stokes equations was proposed in [7], emulating the property of the Navier-Stokes equations that their only nontrivial differential consequence is the Poisson equation for pressure.

This scheme has been tested on several plane flows, including the Karman street. However, testing on three-dimensional flows, including swirling flows, is of greatest interest. Two families of helical flows were analytically described in [8]. Furthermore, the authors of this paper proposed using these flows to test difference schemes.

We have developed and implemented a method in the Sage computer algebra system that allows testing difference schemes, defined symbolically, on helical flows. In this paper, we present our software and the results of its application to the scheme from [7].

2 Helical flows

A helical flow is a flow for which

$$\operatorname{curl} \vec{v} = k\vec{v},$$

where k is a coefficient characterizing the swirl of the flow.

The ABC solution from Ref. [8, p. 75] is described by explicit expressions

$$\begin{cases} u = m(A\sin(kz) + C\cos(ky)), \\ v = m(B\sin(kx) + A\cos(kz)), \\ w = m(C\sin(ky) + B\cos(kx)), \\ p = p_0 - \dfrac{u^2 + v^2 + w^2}{2}, \end{cases}$$

where

$$m = e^{-tk^2/Re}$$

These formulas are implemented in Sage in the standard way.

```
var('x,y,z,t')
var('A,B,C,k,Re,p0')
var('dx,dy,dz,dt')
m=exp(-t*k^2/Re)
u = (A*sin(k*z) + C*cos(k*y))*m
v = (B*sin(k*x) + A*cos(k*z))*m
w = (C*sin(k*y) + B*cos(k*x))*m
p = p0 - 1/2*(u^2+v^2+w^2)
```

3 Defining a difference scheme in Sage

To define a difference scheme from [7], you must first define the difference operators:

```
Dt=lambda f: (f.subs(t=t+dt)-f)/dt
Dx=lambda f: (f.subs(x=x+dx)-f.subs(x=x-dx))/2/dx
Dy=lambda f: (f.subs(y=y+dy)-f.subs(y=y-dy))/2/dy
Dz=lambda f: (f.subs(z=z+dz)-f.subs(z=z-dz))/2/dz
Delta=lambda f: (f.subs(x=x+dx)+f.subs(x=x-dx)-2*f)/dx^2 + \
(f.subs(y=y+dy)+f.subs(y=y-dy)-2*f)/dy^2 + \
(f.subs(z=z+dz)+f.subs(z=z-dz)-2*f)/dz^2
```

It is easy to see that these formulas repeat formula (30) from Ref. [7] in the Sage language. The difference scheme consists of 5 equations [7, Eqs. (34–35)]. Substituting the ABC solution into the first equation is done as follows:

```
Dx(u)+Dy(v)+Dz(w)
```

and returns 0. This means that the exact solution satisfies the first equation of the difference scheme exactly. This is very good, since this equation expresses the mass coservation law. The problems with satisfying this law when using semi-empirical models are well known.

Substituting the solution into the second expression yields a rather complex symbolic expression F_1:

```
F1=Dt(u) + Dx(u^2) + Dy(u*v) + Dz(u*w) + Dx(p) - 1/Re*Delta(u)
```

We expand it as a power series in $\Delta t, \Delta x, \Delta y, \Delta z$:

```
F1.taylor((dt,0),(dx,0),(dy,0),(dz,0),1).factor()
```

The leading term of the expansion of F_1 can be written as

$$\frac{(C\cos(ky) + A\sin(kz))\Delta t k^4 e^{-\frac{k^2 t}{Re}}}{2Re^2},$$

This expression can be estimated from above as

$$\frac{Mk^4\Delta t}{2Re^2}, \quad M = \sqrt{A^2 + B^2 + C^2}. \tag{2}$$

This shows that the second equation is satisfied up to terms of the order of Δt. This low order of approximation is due to the replacement of the derivative with respect to t by the asymmetric operator D_t.

Similar expressions are obtained for F_2 and F_3. The residual in the last equation of the scheme, which approximates the Poisson equation for pressure, is structured differently. Substituting the ABC solution into it is done in the same way:

```
F4=Dx(Dx(p))+Dy(Dy(p))+Dz(Dz(p))+Dx(Dx(u^2)) \
+ Dy(Dy(v^2)) + Dz(Dz(w^2)) \
+2*(Dx(Dy(u*v)) + Dx(Dz(u*w)) + Dy(Dz(v*w))) \
-1/Re*(Dx(Delta(u)) + Dy(Delta(v)) + Dz(Delta(w)))
```

However, the expansion of F_4 into a series yields the sum of the 4th-order terms with respect to $\Delta x, \Delta y$, and Δz as the leading terms. This sum can be estimated from above as

$$\frac{M^2 k^6}{4}\Delta r^4, \quad \Delta r = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}. \tag{3}$$

When choosing the step size, it is natural to try to make the expressions (2) and (3) small. In this case, the equations approximating the Navier-Stokes equations provide an estimate for the step Δt, and the equation approximating the Poisson equation provides an estimate for Δr.

It should be noted that we are attempting to characterize the error in the numerical determination of the solution by the residual. Recently, in a report by Yu. A. Blinkov [9] the author proposed a very elegant method for recalculating errors based on a known residual.

4 Conclusion

The calculations performed show that computer algebra systems can estimate residuals in the execution of difference scheme equations, defined symbolically, using analytical solutions. Based on these residuals, recommendations can be made for choosing the time and space step.

A number of recent publications [10, 11, 12] have found new families of flows that significantly expand the range of analytical solutions on which we can test the proposed difference schemes.

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