



On integration in quadratures of the first-order ordinary differential equations

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Received 18 November 2025. Published 30 November 2025.

Abstract. The first order differential equations of the form $pdx + qdy = 0$ with algebraic functions p and q of variables x and y are considered. Founded on M. Singer theorem theory of symbolical integration of differential equations is stated with the help of S- and P- Volterra integrals. It is shown that the integrating factor is always P-integral and the integral of the differential equation is S-integral. Calculation of the integrating factor is reduced to the search of an algebraic solution of some quasilinear partial differential equation.

Keywords: computer algebra, symbolic computation, differential equations.

MSC numbers: 68W30, 65L99.

PACS numbers: 02.30.Hq, 02.30.Ik.

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