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On integration in quadratures of the first-order ordinary differential equations

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Abstract. The first order differential equations of the form $pdx + qdy = 0$ with algebraic functions p and q of variables x and y are considered. Founded on M. Singer theorem theory of symbolical integration of differential equations is stated with the help of S- and P- Volterra integrals. It is shown that the integrating factor is always P-integral and the integral of the differential equation is S-integral. Calculation of the integrating factor is reduced to the search of an algebraic solution of some quasilinear partial differential equation.

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1. Introduction

Let a first-order differential equation be given

$$p(x,y)dx + q(x,y)dy = 0, \tag{1}$$

where p and q are algebraic functions of the variables x and y. All solutions of this equation, with the exception of a finite number of singular solutions, form a one-parameter family, which can be described by the equation

$$u(x,y) = C, \tag{2}$$

where C is an arbitrary constant, the function u itself is called the integral of the differential equation (1), and the curves on the xy-plane defined by Eq. (2) are the integral curves of the differential equation. When symbolically integrating Eq. (1), they assume that the dependence of u on x and y can be expressed using arithmetic operations, elementary functions, and, in the worst case, quadratures, and strive to find such an expression. This paradigm was established back in the time of Leibniz [1]; today, symbolic integration is both a main topic of seminars on differential equations and a subject of interest for numerous users of computer algebra systems.

In Leibniz's time, it was not clearly defined to which class of functions the desired integral should belong, and this uncertainty persists to present day. In each specific case, it is more or less clear why a particular symbolic expression is considered an analytical solution to a differential equation, but it is completely unclear what the inability of a particular software package to find an analytical solution means.

The formalization of the concept of symbolic integration goes back to the works of Liouville and D. D. Mordukhai-Boltovskoi [2, 3, 4, 5, 6, 7], however, the concept itself was greatly narrowed. In this article, we take as a basis the version of M. Singer [8], which appeared later than others, but comes closest to the classical paradigm. The peculiarity of our interpretation, presented below in Section 2, is the use of the theory of Volterra P-integrals, which allows making the theory free of the elementary function concept. In the following three Sections, integration in quadratures of a given ordinary differential equation (1) is reduced to finding the algebraic integral of a quasilinear partial differential equation based on Liouville's principle. In this case, it is possible to describe the classical constructions of the theory of differential equations in the language of S- and P-quadratures, and to prove the necessary theorems in a generally accessible language, essentially remaining within the framework of Liouville's methods. This very possibility of presenting M. Singer's theory was pointed out in [9].

2. S- and P-quadratures

A characteristic feature of all theories of integration of differential equations in finite form, going back to the work of Liouville, is fixing the list of elementary functions.

This list consists of transcendental functions, tables of whose values began to be compiled as early as the 17th century [1], and therefore it seems a socio-historical phenomenon [10, 11]. The question of why this list includes the sine function but not, say, the Weierstrass $\wp$-function cannot be answered purely mathematically, but the theory itself can be freed from the concept of elementary functions.

For this purpose, let us consider, in addition to integral sums

$$\sum f(x_i)\Delta x,$$

also integral products

$$\prod(1 + f(x_i)\Delta x).$$

As $\Delta x \to 0$, we will call the limit of the sum the S-integral and denote it as

$$\int f(x)dx.$$

The limit of the product will be called the P-integral and we denote it as

$$\mathrm{P}(1 + f(x))dx,$$

without being able to elongate the letter P in the same way as it is done with the letter S to produce the symbol of integration. The relationship between these integrals is given by the formula

$$\mathrm{P}(1 + f(x))dx = e^{\int f(x)dx},$$

thanks to which the exponential function appears in the theory developed below. Only historical circumstances force us to consider S-integrals simpler than P-integrals: S-integrals appeared in the works of Leibniz, while P-integrals appeared only at the end of the 19th century in the works of Volterra and Schlesinger [12]. Under the integral of a differential equation 18th-century authors implicitly understood an expression that contained, in addition to arithmetic operations, the calculation of these two quadratures; however, instead of the second construction, they were forced to speak of elementary functions.

Definition 1. *We say that the dependence of y on x can be expressed using quadratures if y can be represented as an algebraic function of the variable x and auxiliary functions $\alpha_1, \ldots, \alpha_n$ of the variable x, each of which can be expressed using the quadrature from the previous ones:*

$$\alpha_i = \mathrm{S}\, f_i(x, \alpha_1, \ldots, \alpha_{i-1})dx$$

or

$$\alpha_i = \mathrm{P}[1 + f_i(x, \alpha_1, \ldots, \alpha_{i-1})dx],$$

where f_i is an algebraic function of its arguments. We will call the functions α_i themselves quadratures, specifying their type (S or P) when necessary.

For example, the function x^x can be represented using quadratures, since

$$x^x = \exp(x \ln x) = \mathrm{P}\left[1 + \left(1 + \mathrm{S}\,\frac{dx}{x}\right) dx\right].$$

It may happen that one of the quadratures $\alpha_1, \ldots, \alpha_n$ used to represent the dependence of y on x, say, α_i, is expressed algebraically in terms of $x, \alpha_1, \ldots, \alpha_{i-1}$. In this case, this quadrature can be eliminated from the representation, reducing the number of quadratures used. Therefore, without loss of generality, we can assume that the variable x and the quadratures used are not related by any nontrivial algebraic equation.

Here and below, following Liouville, we use algebraic functions as a subset of multivalued analytic functions. Of course, everything said can be reformulated in terms of field theory by considering the extension of the field $\mathbb{C}(x, y)$ affected by the quadratures [8].

The concepts of P- and S-integrals extend to the case of two or more variables. If the differential 1-form

$$udx + vdy$$

is exact, then the expressions

$$S(udx + vdy) = \int udx + vdy$$

and

$$P(1 + udx + vdy) = e^{\int udx+vdy}$$

are functions of the variables x, y.

Definition 2. *We say that the dependence of z on the variables x, y can be expressed using quadratures if z can be represented as an algebraic function of the variables x, y and auxiliary functions $\alpha_1, \ldots, \alpha_n$ of the variables x, y, each of which can be expressed using the quadrature from the previous ones:*

$$\alpha_i = \mathrm{S}\left(f_i(x, \alpha_1, \ldots, \alpha_{i-1})dx + g_i(x, \ldots)dy\right)$$

or

$$\alpha_i = \mathrm{P}\left[1 + f_i(x, \alpha_1, \ldots, \alpha_{i-1})dx + g_i(x, \ldots)dy\right],$$

where f_i, g_i are algebraic functions of their arguments.

For example, the function x^y can be represented using quadratures, since

$$x^y = \exp(y \ln x) = \exp \int (\frac{ydx}{x} + \ln xdy) =$$
$$= \mathrm{P}\left[1 + \frac{ydx}{x} + \mathrm{S}\left(\frac{dx}{x}\right) \cdot dy\right].$$

Definition 3. *We say that differential equation* (1) *is integrable by* n *quadratures if it has a one-parameter family of integral curves defined by the equation*

$$F(\alpha_n, \alpha_{n-1}, \dots, x, y, C) = 0, \tag{3}$$

the left-hand side of which is an algebraic function of x, y *and quadratures* $\alpha_1, \dots, \alpha_n$.

Example 1. *The first symbolic integrator, J. Moses's Soldier [13] , was based on the following observation: Eq.* (1) *admits an integrating factor depending only on* x *if and only if*

$$r = \frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right)$$

is independent of y. *In this case, the integrating factor is a P-quadrature:*

$$\mu = e^{\int r(x)dx} = \mathrm{P}(1 + r(x)dx),$$

and the integral of the differential equation can be expressed using two quadratures:

$$u = \int \mathrm{P}(1 + rdx) \cdot (pdx + qdy).$$

Integral curves are given by the equation

$$\alpha_2 = C,$$

where

$$\alpha_2 = \int \alpha_1 \cdot (pdx + qdy), \quad \alpha_1 = \mathrm{P}(1 + rdx).$$

Thus, in this case, F *is a linear function of the constant* C *and the last second quadrature. The number of quadratures required to represent integral curves can be reduced, for example, when the integral* $\mathrm{P}(1+rdx)$ *is evaluated in algebraic functions of the variable* x.

Definition 3 allows accurate formulation of the problem of integration in quadratures.

Problem 1. *For a given differential equation* (1), *determine whether it can be integrated using a finite number of quadratures. If so, write out a representation of the integral using quadratures.*

Creating an algorithm for solving this problem is the ultimate goal of the theory of symbolic integration of differential equations within the classical paradigm.

3. Liouville's principle

The integrating factor can contain quadratures, which is very strange from the point of view of calculating constants, since each quadrature is defined up to a constant.

Example 2. *Let us return to Example 1, where the integral curves are given by the equation*

$$\int e^{\int r dx} \cdot (p dx + q dy) = C.$$

The left-hand side contains two quadratures, and therefore two constants, and at the same time, by Cauchy's theorem, the family of integral curves must be one-parameter. One assertion does not contradict the other, since the equation

$$\int e^{\int r dx + C_1} \cdot (p dx + q dy) + C_2 = C$$

can be reduced to the form

$$\int e^{\int r dx} \cdot (p dx + q dy) = (C - C_2) e^{-C_1},$$

which differs from the original only by the constant.

In general, such "variations of constants" should not increase the number of independent constants, which imposes very significant conditions on the form of Eq. (3). This technique was first used by Liouville in his work on the integration of elementary functions [14], and is now used in various theories of integration of differential equations [3, 11]. We formulate it as the following theorem.

Theorem 1 (Liouville's principle). *Let (i) the equation*

$$F(\alpha_n, \alpha_{n-1}, \dots, x, y, C) = 0$$

describe the integral curves of a differential equation, (ii) the transcendental functions $\alpha_1, \dots, \alpha_n$ of the variables x and y satisfy some algebraic system of differential equations (S) that allows expressing all partial derivatives of these functions as algebraic functions of $\alpha_n, \alpha_{n-1}, \dots, x, y$, and (iii) reducing the number of transcendental functions required to represent the integral curves be impossible. Then the curves

$$F(\beta_n, \beta_{n-1}, \dots, x, y, C) = 0, \tag{4}$$

where $\beta_n, \dots$ is any other solution of system (S), are also integral.

If the functions $\alpha_1, \dots, \alpha_n$ are quadratures, then the i-th equation of system (S) has the form

$$d\alpha_i = g_i(\alpha_{i-1}, \dots, y, x) dx + h_i(\alpha_{i-1}, \dots, y, x) dy$$

or

$$\frac{d\alpha_i}{\alpha_i} = g_i(\alpha_{i-1},\dots,y,x)dx + h_i(\alpha_{i-1},\dots,y,x)dy$$

depending on whether α_i is S- or P-quadrature.

For example, for the two transcendental functions that appear when integrating the differential equation from Example 1, system (S) will be

$$d\alpha_2 = \alpha_1 \cdot (pdx + qdy), \quad \frac{d\alpha_1}{\alpha_1} = rdx.$$

By Liouville's principle, the expression

$$\int e^{\int r(x)dx + C_1} (pdx + qdy) + C_2$$

is also an integral of a differential equation.

Proof. Equation (4) defines a family of integral curves if and only if the 2-form

$$dF(\alpha_n, \alpha_{n-1}, \dots, x, y, C) \wedge (pdx + qdy)$$

vanishes on the manifold (4). Since $\beta_1,\dots$ satisfy the system of differential equations (S), by the second condition of the theorem, the partial derivatives of F with respect to x and y, that is,

$$\frac{\partial F}{\partial \beta_n}\frac{\partial \beta_n}{\partial x} + \dots + \frac{\partial F}{\partial x} \quad \text{and} \quad \frac{\partial F}{\partial \beta_n}\frac{\partial \beta_n}{\partial y} + \dots + \frac{\partial F}{\partial y}$$

are algebraic functions of x, y, and transcendental functions $\beta_1,\dots,\beta_n$. Substituting these expressions into 2-form, we obtain

$$G(\beta_n, \beta_{n-1}, \dots, x, y, C) dx \wedge dy.$$

The condition for the expression G to vanish on the manifold (4) can be rewritten as a system (G) of algebraic equations relating x, y and $\beta_1,\dots,\beta_n$. This system is satisfied if we substitute the functions $\alpha_1,\dots$ for $\beta_1,\dots,$. By condition 3.), there can be no nontrivial relations between x, y and the functions $\alpha_1,\dots$. Therefore, system (G) is trivial, and the 2-form written above vanishes on the entire manifold (4) for any choice of $\beta_1,\dots$. □

4. Integral of a differential equation

The Liouville principle allows significant simplification of the possible form of the equation of integral curves.

Lemma 1. *If a differential equation is integrated using n quadratures, then it has a one-parameter family of integral curves, which are described by an equation of the form*

$$\int g(\alpha_{n-1},\ldots,y,x)dx + h(\alpha_{n-1},\ldots,y,x)dy = c. \tag{5}$$

Proof. Let us consider the two possible cases separately.

(i) Let α_n be an S-integral, then

$$\alpha_n = \int g(\alpha_{i-1},\ldots,y,x)dx + h(\alpha_{i-1},\ldots,y,x)dy$$

and for any constant value of ϵ

$$\beta_n = \alpha_n + \epsilon, \quad \beta_{n-1} = \alpha_{n-1},\ldots$$

is also a solution to the system (S) discussed in Liouville's principle. Therefore, the equation

$$F(\alpha_n + \epsilon, \alpha_{n-1},\ldots,y,x,c) = 0$$

defines a certain family of integral curves of the original differential equation, depending on two parameters c and ϵ. This family, however, cannot be two-parameter, so we obtain almost all integral curves if we fix c and leave ϵ variable. In this case, the equation

$$F(\alpha_n + c, \alpha_{n-1},\ldots,y,x) = 0$$

also defines a one-parameter family of integral curves. Solving this equation for the first argument, we obtain

$$\alpha_n + c = f(\alpha_{n-1},\ldots,y,x)$$

or

$$\int g(\alpha_{i-1},\ldots,y,x)dx + hdy - df = \text{const},$$

which is what the lemma states.

(ii) Let α_n be a P-integral, then

$$\alpha_n = \exp\int g(\alpha_{i-1},\ldots,y,x)dx + h(\alpha_{i-1},\ldots,y,x)dy$$

and for any constant value of ϵ

$$\beta_n = \epsilon\alpha_n, \quad \beta_{n-1} = \alpha_{n-1},\ldots$$

is also a solution to the system (S) discussed in Liouville's principle. Therefore, the equation

$$F(\epsilon\alpha_n, \alpha_{n-1},\ldots,y,x,c) = 0$$

defines a certain family of integral curves of the differential equation depending on two parameters c and ϵ, and

$$F(c\alpha_n, \alpha_{n-1}, \dots, y, x) = 0$$

is a one-parameter family of integral curves. Solving this equation for the first argument, we obtain

$$c\alpha_n = f(\alpha_{n-1}, \dots, y, x)$$

or

$$\exp \int g(\alpha_{i-1}, \dots, y, x)dx + hdy - \frac{df}{f} = \text{const.}$$

Taking logarithm, we obtain the assertion of the lemma. □

Thanks to Lemma 1, the concept of the integral of a differential equation acquires a meaning that could not be precisely expressed in the 18th century: on the left-hand side of the equality (5) is the S-integral, which is a function of the variables x and y, constant on any integral curve, i.e., the integral of the differential equation. Therefore, the lemma itself can be formulated as follows: if a differential equation is integrated using quadratures, then one of the integrals of this differential equation is an S-integral.

5. Integrating factor of a differential equation

Further, without loss of generality, we can assume that the family of integral curves is described by a quadrature

$$\int g(\alpha_{n-1}, \dots, y, x)dx + h(\alpha_{n-1}, \dots, y, x)dy = c.$$

The equation

$$gdx + hdy = 0$$

defines the same relationship between differentials as the original equation $pdx + qdy = 0$, therefore

$$\frac{dy}{dx} = -\frac{g}{h} = -\frac{p}{q}$$

and, therefore, there exists a function μ of variables x and y such that

$$gdx + hdy = \mu \cdot (pdx + qdy),$$

which, by definition, is an integrating factor of the form $pdx + qdy$. Since

$$\mu = \frac{g(\alpha_{n-1}, \dots, y, x)}{p(x, y)},$$

the factor is an algebraic function of $\alpha_{n-1}, \dots, y, x$.

In textbooks, it is common to note that the problem of finding an integrating factor is no simpler than the problem of integrating a differential equation. However, if a family of curves can be represented using n quadratures, then the integrating factor can be represented using $n-1$ quadratures, and therefore the problem of finding the factor should be simpler than the original problem. The following theorem clarifies the matter quite well.

Theorem 2. *If a differential equation is integrated using a finite number of quadratures, then the integrating factor is either an algebraic function of x, y or a P-quadrature of some exact differential form whose coefficients are algebraic functions of the variables x, y.*

Proof. (i) If a family of integral curves can be described using algebraic functions, that is, by an equation of the form

$$f(x, y) = c$$

where f is an algebraic function of its arguments, then the form $pdx + qdy$ has an integrating factor among the algebraic functions. If a single quadrature is required to describe this family, then by Lemma 1 this family can be defined using the integral

$$\int g(y, x)dx + h(y, x)dy = c,$$

in this case, the factor is equal to g/p and is therefore again an algebraic function of x, y

(ii) Now assume that the family of curves can be represented using $n > 1$ quadratures, and this number cannot be reduced. In this case, the factor is representable using $n-1$ quadratures, that is,

$$\mu = \mu(\alpha_{n-1}, \dots, y, x),$$

where the right-hand side is an algebraic function of its arguments. We will prove that in this case the integrating factor is a P-quadrature.

Case 1. If the last quadrature α_{n-1} is an S-quadrature, then, by Liouville's principle, the expression

$$\mu = \mu(\alpha_{n-1} + \epsilon, \dots, y, x)$$

for any constant value of ϵ is an integrating factor. By Euler's theorem, the ratio of two integrating factors is either the integral of differential equation or simply a constant. The ratio

$$\frac{\mu(\alpha_{n-1} + \epsilon, \dots, y, x)}{\mu(\alpha_{n-1}, \dots, y, x)}$$

cannot be an integral of differential equation, since in that case the family of integral curves could be described using $n-1$ quadratures. Therefore, this ratio is independent of x, y and quadratures:

$$\mu(\alpha_{n-1}+\epsilon,\dots,y,x) = f(\epsilon)\mu(\alpha_{n-1},\dots,y,x).$$

By virtue of the well-known theorem on functional equations, μ in this case is an exponential function of α_{n-1}, which is impossible.

Case 2. If the last quadrature α_{n-1} is a P-quadrature, then, for the same reasons as above, the ratio

$$\frac{\mu(\alpha_{n-1},\dots,y,x)}{\mu(\alpha_{n-1},\dots,y,x)}$$

is independent of x, y and quadratures, that is,

$$\mu(\epsilon\alpha_{n-1},\dots,y,x) = f(\epsilon)\mu(\alpha_{n-1},\dots,y,x).$$

This is only possible if μ is a linear homogeneous function of α_n:

$$\mu = j(\alpha_{n-2},\dots,y,x)\cdot\alpha_{n-1}.$$

Since α_{n-1} is a P-quadrature, it can be represented as

$$\alpha_{n-1} = \exp\left(\int g_{n-1}(\alpha_{n-2},\dots)dx + h_{n-1}dy\right),$$

Therefore, the original differential equation admits a factor of the form

$$\mu = \exp\left(\int g_{n-1}(\alpha_{n-2},\dots)dx + h_{n-1}dy + \frac{dj}{j}\right),$$

that is, μ is a P-quadrature.

(iii) If $n=2$, then the assertion of the theorem follows immediately from (ii). We prove by contradiction that the case $n>2$ is impossible. To this end, we now assume that the family of curves is representable using $n>2$ quadratures, and this number cannot be reduced. By (ii), the factor is a P-quadrature, that is,

$$\mu = e^{\int g(\alpha_{n-2},\dots,x)dx + h(\alpha_{n-2},\dots,x)dy}.$$

where the right-hand side of the integral is an algebraic function of its arguments.

If the last quadrature α_{n-2} is an S-quadrature, then by Liouville's principle the expression

$$\mu = e^{\int g(\alpha_{n-2}+\epsilon,\dots,x)dx + h(\alpha_{n-2}+\epsilon,\dots,x)dy}$$

is also an integrating factor, and the ratio of two such factors is either an integral of differential equation or a constant. In the first case, we obtain that the expression

$$\int [g(\alpha_{n-2}+\epsilon,\dots,x)-g(\alpha_{n-2},\dots,x)]dx$$
$$+\,[h(\alpha_{n-2}+\epsilon,\dots,x)-h(\alpha_{n-2},\dots,x)]dy$$

are integrals of differential equation, which contradicts the assumption that the number n in the representation of integral curves through quadratures cannot be reduced. In the second case, g and h are independent of α_{n-1}, and therefore

$$\mu = e^{\int g(\alpha_{n-3},\dots,x)dx+h(\alpha_{n-3},\dots,x)dy}.$$

In this case, the integral curves of the differential equation can be represented using $n-1$ quadratures, which is also impossible.

A similar contradiction arises when α_{n-2} is a P-quadrature. □

The proven theorem allows us to significantly strengthen the assertion of the lemma 1.

Theorem 3. *If the differential equation $pdx + qdy = 0$ is integrated using a finite number of quadratures, then one of the integrals of this differential equation is an S-integral and its representation requires no more than two quadratures.*

The restriction on n significantly distinguishes this theory from classical Galois theory, in which the length of the sequence of radicals required to represent the solution is unlimited.

6. Algebraization of the problem of integrating a differential equation

By Theorem 2, the integrating factor can always be considered a P-integral, that is,

$$\mu = \mathrm{P}(1 + udx + vdy) = \exp \int udx + vdy, \tag{6}$$

where $udx + vdy$ is the exact differential form with coefficients u and v depending on x and y algebraically. To consider all possible cases together, we will not assume that μ is necessarily a transcendental function, and in the case where μ is an algebraic function of x and y, we write it as an integral

$$\mu = \exp \int \frac{d\mu}{\mu} = \mathrm{P}\left(1 + \frac{d\mu}{\mu}\right).$$

Theorem 4. *For the expression (6) to be an integrating factor of the differential equation (1), it is necessary and sufficient that the equation*

$$q^2\frac{\partial v}{\partial x} - pq\frac{\partial v}{\partial y} - vq^2\frac{\partial}{\partial y}\frac{p}{q} - q^2\frac{\partial}{\partial y}\frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0 \tag{7}$$

with respect to the variable v has a solution in algebraic functions of the variables x and y.

Proof. For the expression (6) to be an integrating factor of the differential equation (1), it is necessary and sufficient that both forms

$$udx + vdy \quad \text{and} \quad \mu(pdx + qdy)$$

be exact. The first condition yields

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \tag{8}$$

and the second

$$p\frac{\partial \mu}{\partial y} - q\frac{\partial \mu}{\partial x} + \mu\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0. \tag{9}$$

After substituting the expression for the factor (6) into equation (9), we obtain

$$pv - qu + \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = 0. \tag{10}$$

Eliminating the unknown u from equations (8) and (10), we obtain the equation

$$\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\frac{1}{q}\left(pv + \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0,$$

which, after simplifications, becomes equation (7). □

Example 3. *Equation (7) has a trivial solution $v = 0$ if and only if when*

$$\frac{\partial}{\partial y}\frac{1}{q}\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right) = 0;$$

in this case,

$$\mu = \exp\int u(x)dx.$$

This is exactly the expression obtained for the multiplier in example 1.

The proven Theorem 4 reduces the question of integration by quadratures of a given differential equation to the integration of Eq. (7) in algebraic functions, that is, it completely eliminates any transcendental functions from the problem. Let us formulate this problem.

Problem 2. *For a given quasilinear partial differential equation*

$$p\frac{\partial v}{\partial x} + q\frac{\partial v}{\partial y} + r = 0,$$

where p, q, r are algebraic functions of the variables x, y, find out whether it has an algebraic solution and, if so, write out the solution.

Investigating the algorithmic solvability of this purely algebraic problem is extremely important for symbolic integration. At the beginning of the 20th century, M.N. Lagutinski proposed a method for finding such solutions in the case where the degrees of the monomials included in the desired solution v are bounded from above by a given constant N [15, 16, 17, 18, 19, 20]. This method is currently implemented in Sage [21] and has been successfully tested on A.F. Filippov's problem book [22]. At the same time, it is not clear whether fixing the constant N is required for the algorithmic solvability of problem 2. It also seems important to clarify how, in the case of equation (7), the value N is related to the parameters characterizing the integral curves of equation (1).

Theorem 4 was cited in [23], where several examples of its use for calculating integrals in the Sage computer algebra system are also given.

7. Conclusion

The theory developed above allows us to quite simply describe the structure of solutions to differential equations solvable in quadratures. If a family of integral curves of a differential equation (1) can be described using an equation containing arithmetic operations, as well as S- and P-quadratures, then this equation admits:

- According to Theorem 2, the integrating factor μ is among P-integrals of exact differential forms whose coefficients depend algebraically on the variables x and y,
- According to Theorem 3, the integral u is among S-integrals.

This is consistent with the classical concepts that arose in the 18th century when solving specific examples: the integral of a differential equation is always an S-integral or quadrature, the integrating factor is always simpler than the integral, etc. The classical paradigm is not “non-rigorous”, but it only concerns equations solvable in quadratures.

Theorem 4 reduces the problem 1 of integrating a given differential equation in quadratures to a more general and at the same time purely algebraic problem (2). Thus, to solve the classical problem of integrating differential equations in quadratures, it seems important to create algorithms for solving Problem 2 using computer algebra.

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