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On the solution of the Helmholtz equation in an ellipsoid of revolution with a singularity at the focus

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Abstract. The propagation of a monochromatic spherical wave emitted from one of the foci of an ellipsoid and reflected at its boundary is considered. From the point of view of geometric optics, the second focus of the ellipsoid should be a singularity of the solution, since the wave reflected from the ellipsoid boundaries "converges" there. In wave optics, on the contrary, no singularity can exist there. The corresponding boundary value problem is formulated within the framework of wave optics and solved using the finite element method. Computer experiments were performed in FreeFem++, using a weak formulation of the problem in a spherical coordinate system. A series of computer experiments demonstrate the changes in this wave with increasing frequency. It is shown that, starting from a certain frequency, a clearly expressed extremum appears in the region of the second focus.

Keywords: finite elements, boundary value problems, focus

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1 Introduction

Consider an ellipsoid of revolution V. The simplest wave model describing waves in such an ellipsoid is given by the boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0, \\ u|\partial V = 0. \end{cases} \tag{1}$$

Place a spherical wave source at one of its foci

$$u' = -\frac{e^{ikr}}{r}$$

and seek a solution of the problem (1) in the form

$$u = u' + u'',$$

where the term u'' describes the response of the ellipse to the spherical wave. The response satisfies the problem

$$\begin{cases} \Delta u'' + k^2 u'' = 0, \\ u''|\partial V = -u'. \end{cases} \tag{2}$$

This problem has a unique solution if k differs from the resonant values [1, §5.7.2]. Thus, within the framework of wave theory, the response can and should be considered regular.

From the point of view of geometric optics, the response will be a spherical wave with a second focus as its center, that is, an internal singularity. For theoretical reasons, linking geometric optics and wave optics in this case is extremely difficult, since the effect depends on an infinite number of transformations, and the focal point itself is singular.

We will attempt to use direct calculations to understand what happens to the regular response in the vicinity of the second focus and thereby establish a connection between the wave and geometric theories.

2 Imaginary part of the response

The imaginary part of the response can be written explicitly. Indeed, the original wave is equal to

$$u' = -\frac{e^{ikr}}{r} = -\frac{\cos kr}{r} - i\frac{\sin kr}{r}.$$

The imaginary part has no singularity at zero, and therefore the solution to the problem

$$\begin{cases} \Delta v + k^2 v = 0, \\ v|\partial V = \frac{\sin kr}{r} \end{cases}$$

is inevitably

$$v = \frac{\sin kr}{r}.$$

This means that the imaginary part of the response is equal, up to the sign, to the imaginary part of the original wave, and $u = u' + u''$ is real. This reasoning is not applicable to the real part, since

$$-\frac{\cos kr}{r}$$

has a singularity at zero, while u'' does not.

3 On the numerical calculation of response

The numerical solution of the problem (2) is conveniently performed using the finite element method [2, 3], which has proven itself well in problems of electrodynamics [4]. We will use a free implementation of this method, the FreeFem++ system [5].

We will consider the ellipsoid of revolution in a spherical coordinate system, placing its origin at the focus from which the spherical wave originates. The ellipsoid is then described by the inequalities

$$0 < r < \frac{p}{1 - e\cos\theta}, 0 < \theta < \pi, 0 < \phi < 2\pi.$$

To use the FreeFem++ system, it is necessary to rewrite the Helmholtz problem in a generalized form and in a spherical coordinate system. Due to symmetry, the solution to Helmholtz problem is independent of ϕ and is reduced to solving the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial u''}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial u''}{\partial\theta} + k^2 u'' = 0$$

in a curvilinear trapezoid

$$0 < r < \frac{p}{1 - e\cos\theta}, \quad 0 < \theta < \pi.$$

We define an arbitrary smooth function v, multiply the last equation by $vr^2 \sin\theta$ and integrate over the trapezoid, then we obtain

$$\iint\limits_V \left(v\sin\theta \frac{\partial}{\partial r} r^2 \frac{\partial u''}{\partial r} + v\frac{\partial}{\partial\theta}\sin\theta\frac{\partial u''}{\partial\theta} + k^2 v u'' r^2 \sin\theta \right) d\theta dr = 0.$$

Integrating by parts, we transform the left-hand side to a sum of integrals over a trapezoid, namely,

$$-\iint\limits_V \left(r^2 \sin\theta \frac{\partial v}{\partial r}\frac{\partial u''}{\partial r} + \sin\theta\frac{\partial v}{\partial\theta}\frac{\partial u''}{\partial\theta} + k^2 v u'' r^2 \sin\theta \right) d\theta dr$$

and integrals over the boundary of the trapezoid

$$\int\limits_{\partial V} \left(n_r r^2 \frac{\partial u''}{\partial r} + n_\theta \frac{\partial u''}{\partial \theta} \right) v \sin\theta ds.$$

If u'' does not increase to infinity on the boundary, then on the sections $\theta = 0$ and $\theta = \pi$, the integral over the boundary is zero due to $\sin\theta = 0$, and on the section $r = 0$, ”— due to $n_\theta = 0$. Therefore, only a boundary segment remains

$$C: \quad r = \frac{p}{1 - e\cos\theta}, \quad 0 < \theta < \pi,$$

on which we must set the boundary condition

$$u'' = u'.$$

When using the finite element method, this is achieved by setting a boundary condition of the third kind

$$n_r r^2 \frac{\partial u''}{\partial r} + n_\theta \frac{\partial u''}{\partial \theta} + h(u'' - u') = 0,$$

taking a very large h. As a result, we obtain the following weak statement of the problem: find a function $u \in W_2^1(V)$ such that

$$\iint\limits_V \left(r^2 \sin\theta \frac{\partial v}{\partial r}\frac{\partial u''}{\partial r} + \sin\theta \frac{\partial v}{\partial \theta}\frac{\partial u''}{\partial \theta} + k^2 v u'' r^2 \sin\theta \right) d\theta dr +$$
$$+ \int\limits_C h(u'' - u') v \sin\theta ds = 0 \quad \forall v \in W_2^1(V).$$

This problem can be easily solved using FEM in FreeFem++.

4 Computer experiment results

We took $p = 1$, $e = \frac{1}{2}$ and $h = 10^{-5}$, verifying that the solution remains unchanged as h increases further.

In the figures below, Figs. 1–15, the abscissa axis is θ, and the ordinate axis is r. The first focus is broken into the line $r = 0$ due to the transition to a spherical coordinate system. The second focus lies on the $\theta = 0$ axis at a distance from the upper edge of the region equal to the width of the side $\theta = \pi$.

For small k, noticeable humps appear on the ellipsoid axis: first one ($k = 2$), then two ($k = 5$), then three ($k = 6$). Then comes the range $6 < k < 10$, which is difficult to describe in words. At $k = 10$, a clear and very noticeable extremum appears in the region of the second focus. With further increase in k, this extremum does not shift anywhere, but becomes increasingly sharper. At $k = 13$, the structure of the spherical wave emitted from the second focus is clearly visible. It should also be noted that the position of the extremum lies slightly higher than the actual focus.

For the same k but an oval other than an ellipsoid, the response has no such an extremum. Sometimes the extremum center is in focus, and sometimes it is not.

As the eccentricity increases, the effect increases, but significantly more finite elements must be used. Fig. 18 clearly shows a diverging wave emitted from the second focus already at $k = 10$.

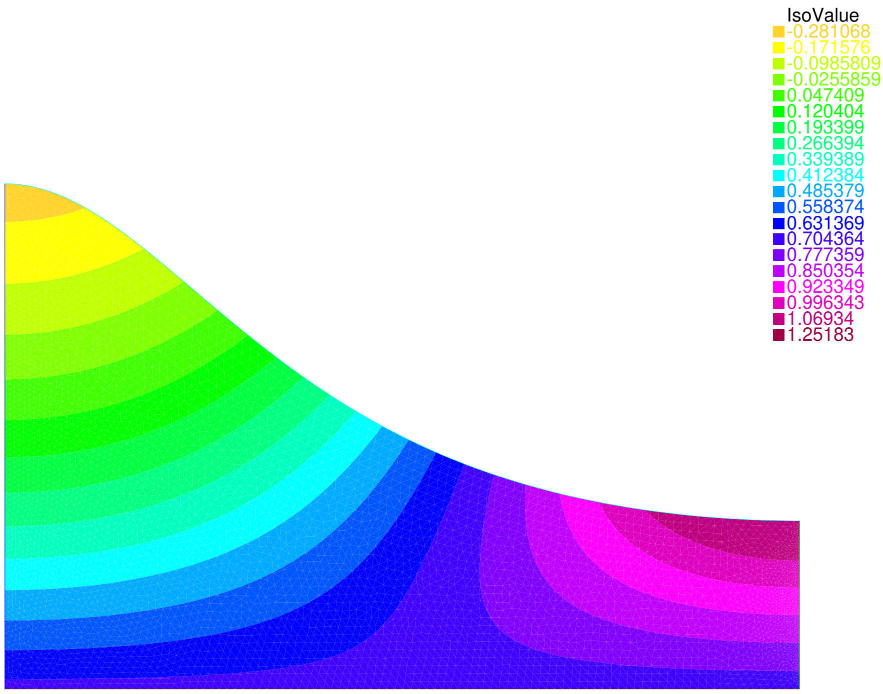

Figure 1: Response plot for $k = 1$

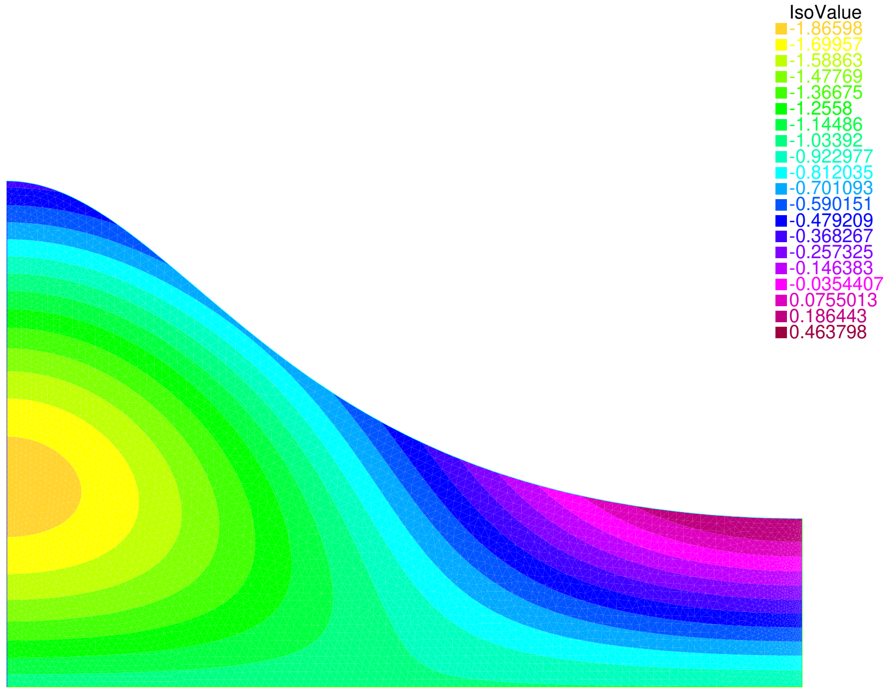

Figure 2: Response plot for $k = 2$

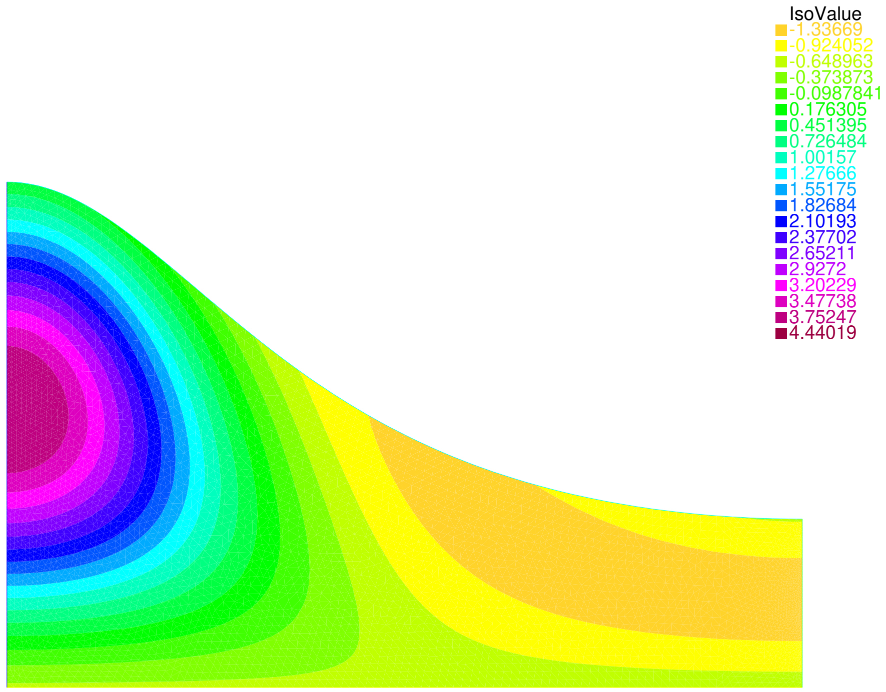

Figure 3: Response plot for $k = 3$

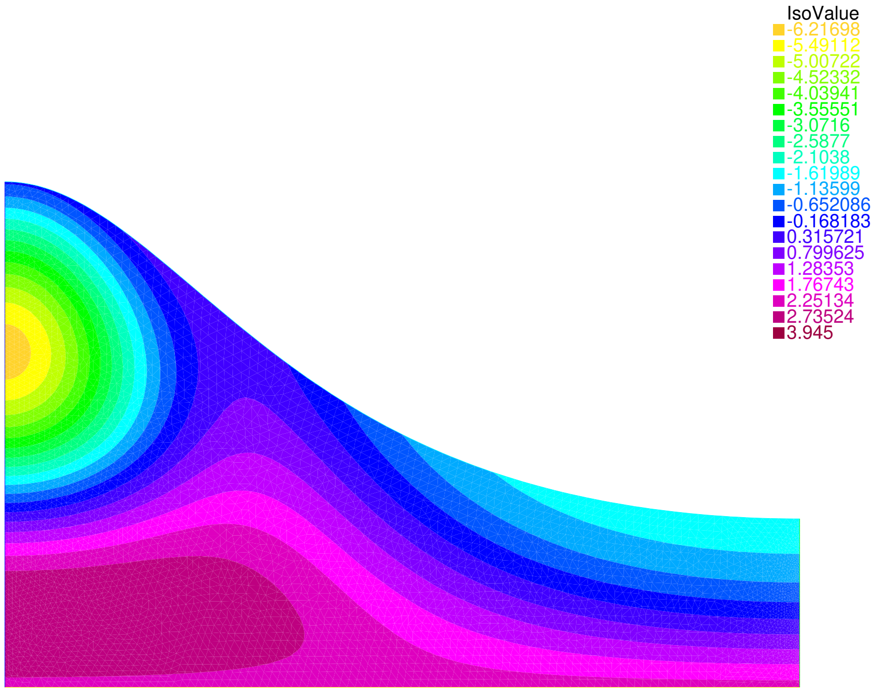

Figure 4: Response plot for $k = 4$

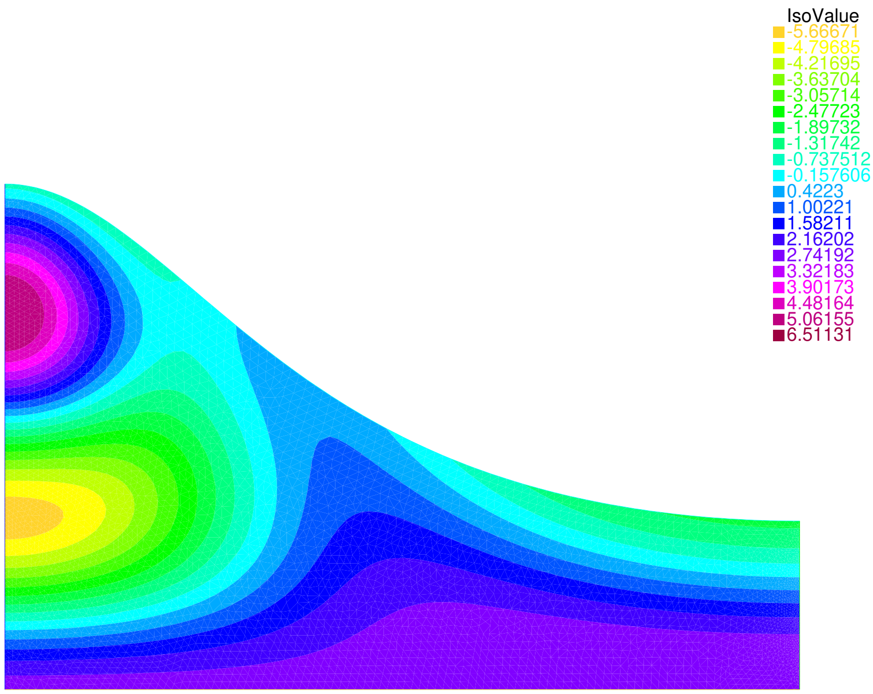

Figure 5: Response plot for $k = 5$

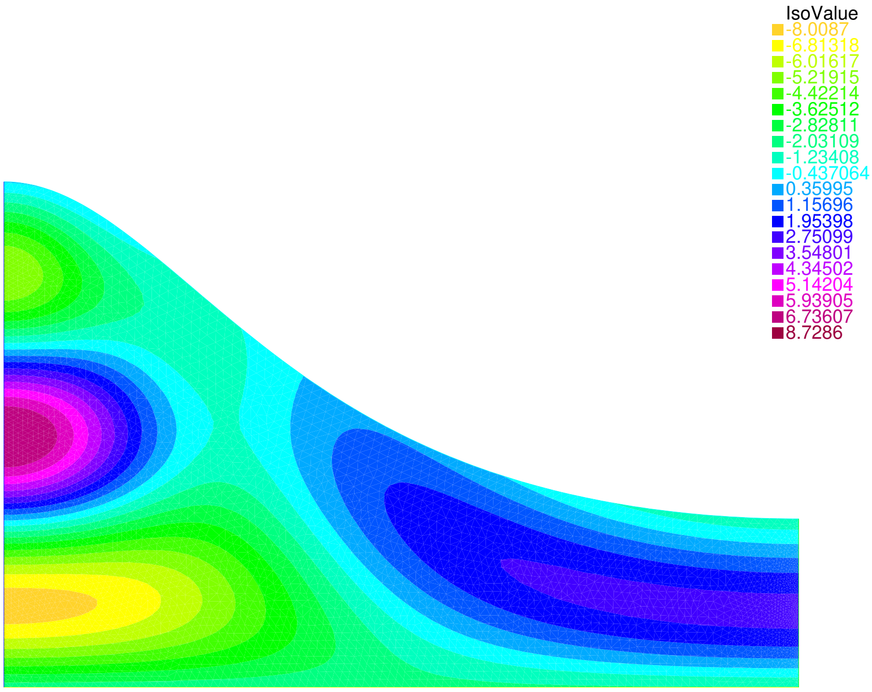

Figure 6: Response plot for $k = 6$

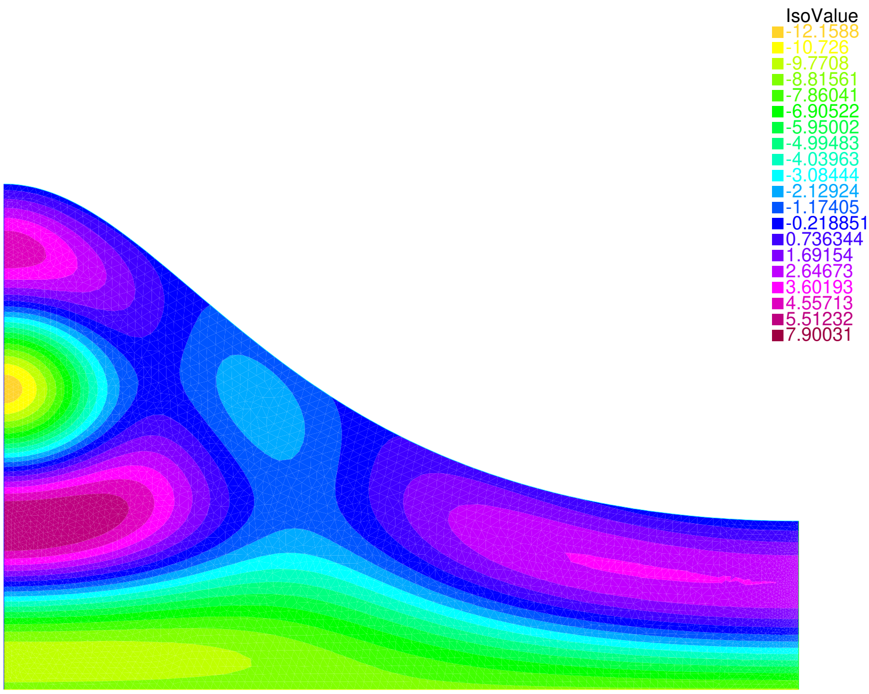

Figure 7: Response plot for $k = 7$

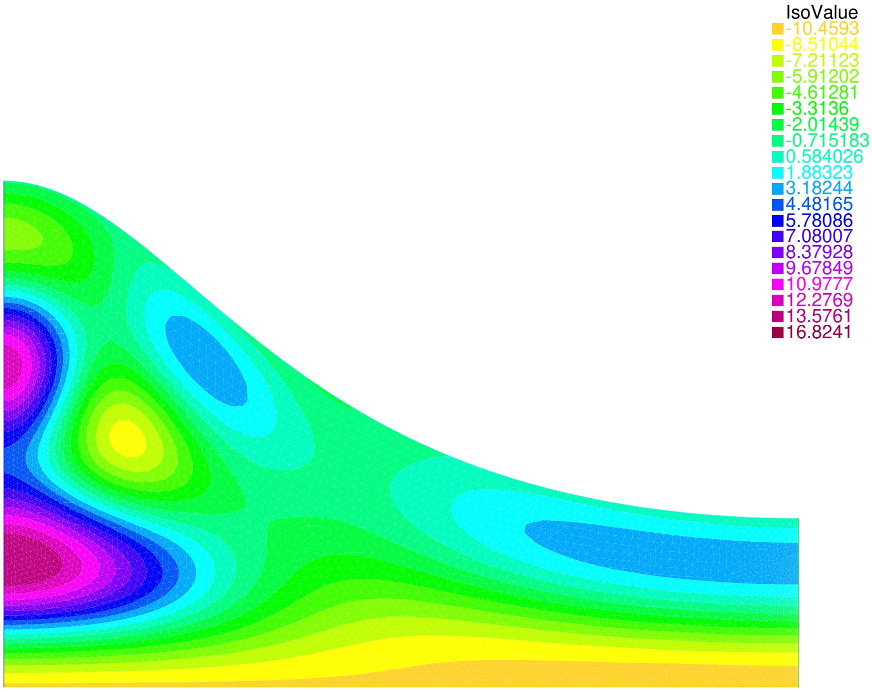

Figure 8: Response plot for $k = 8$

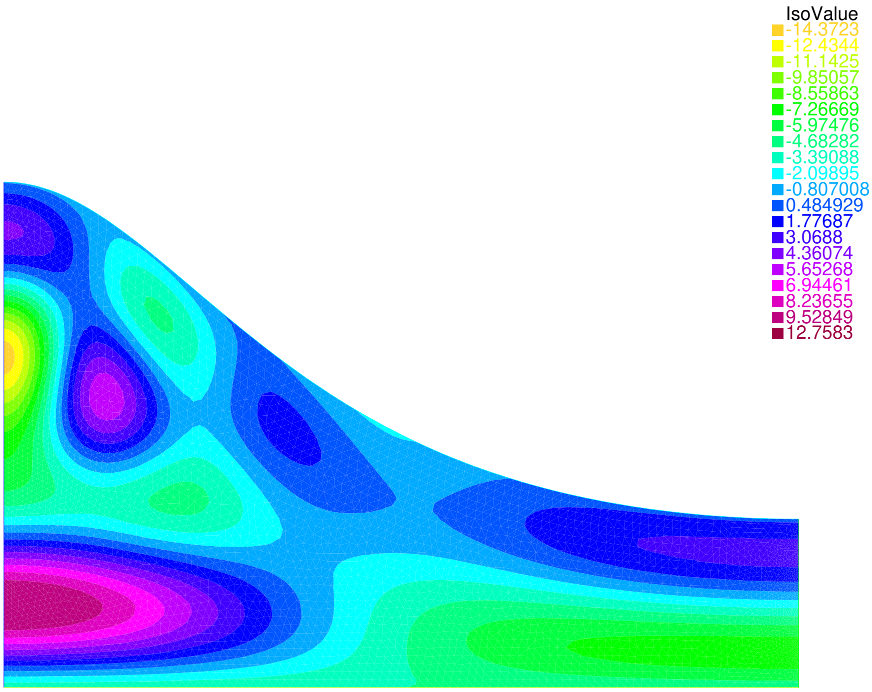

Figure 9: Response plot for $k = 9$

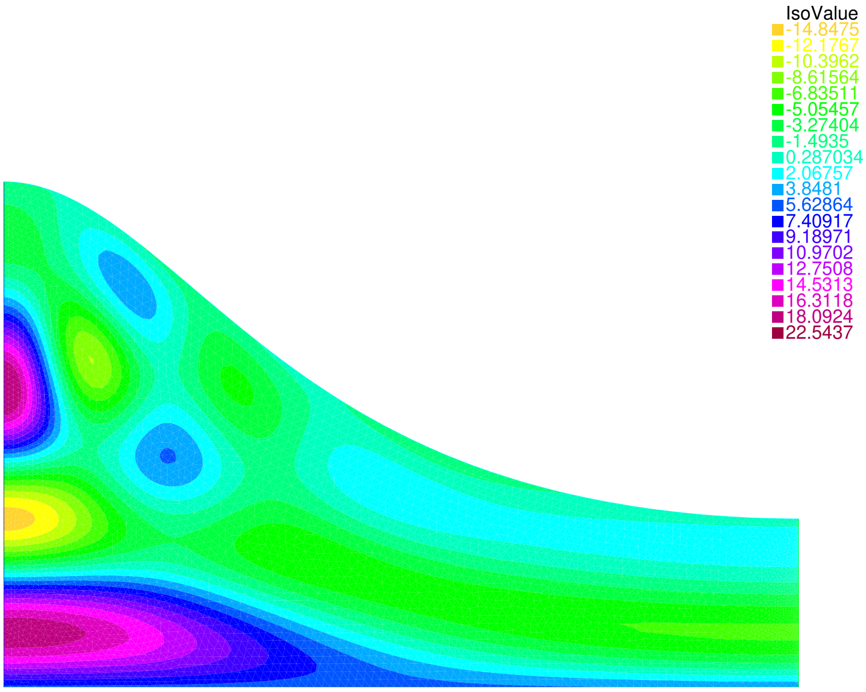

Figure 10: Response plot for $k = 10$

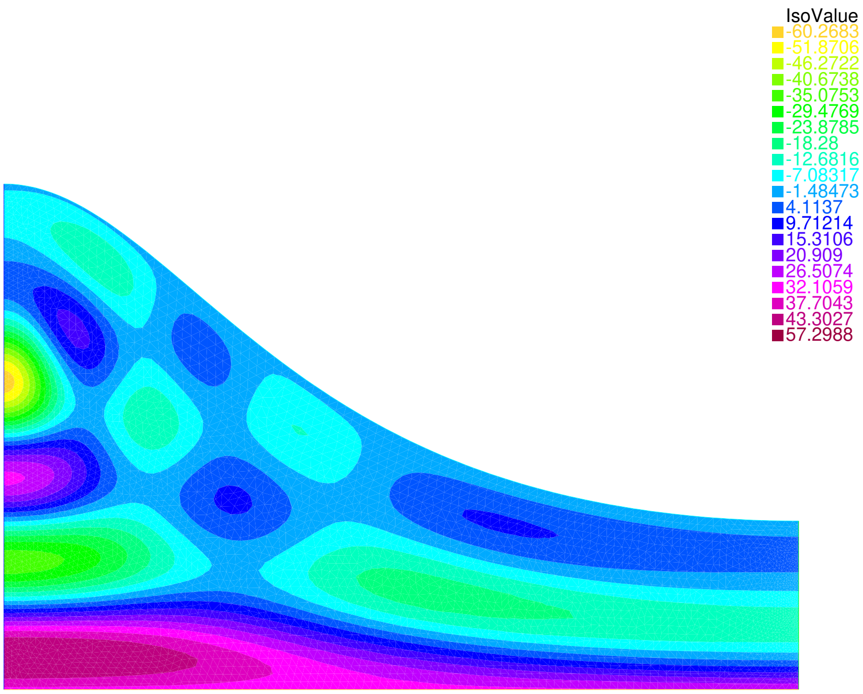

Figure 11: Response plot for $k = 11$

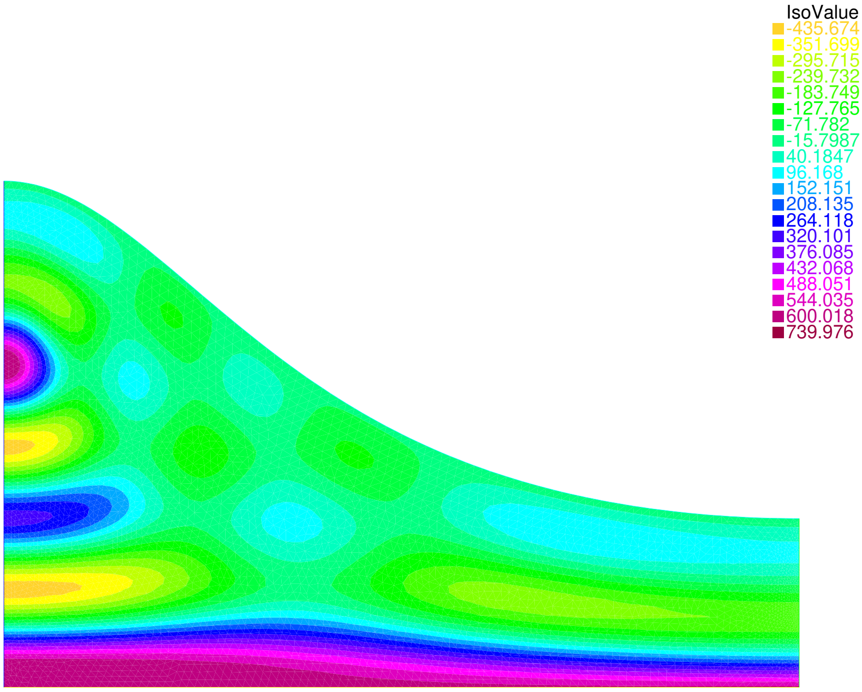

Figure 12: Response plot for $k = 12$

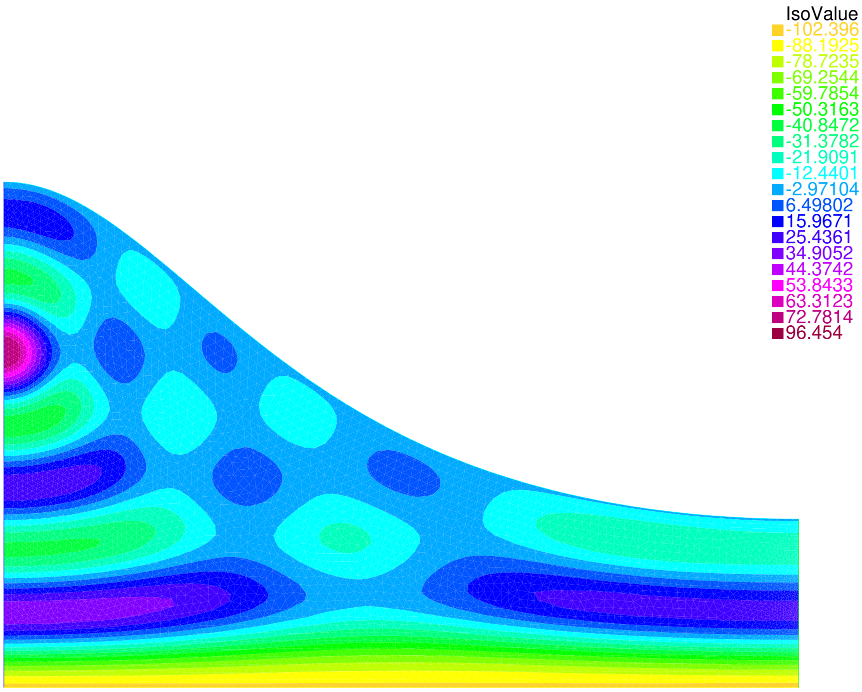

Figure 13: Response plot for $k = 13$

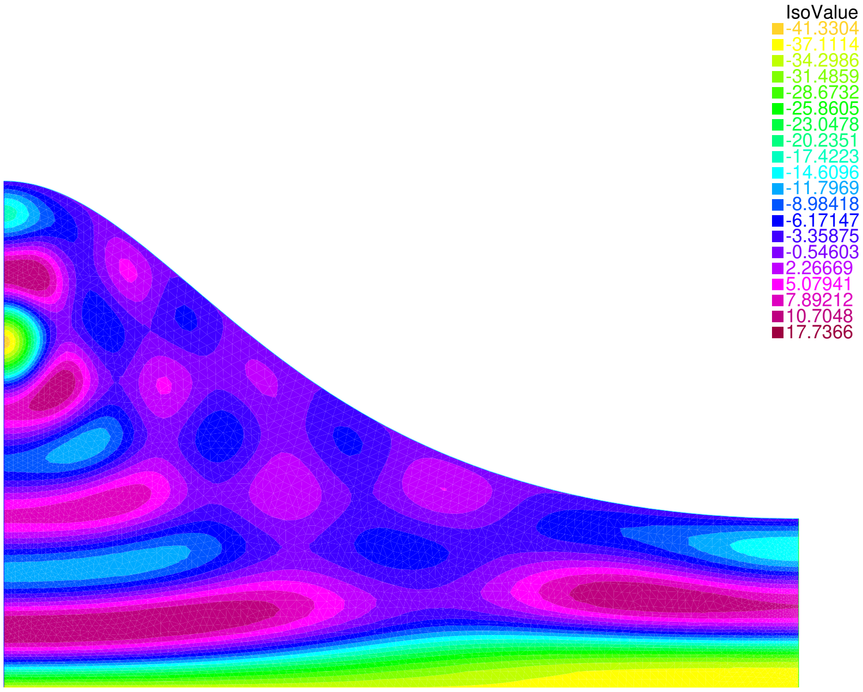

Figure 14: Response plot for $k = 14$

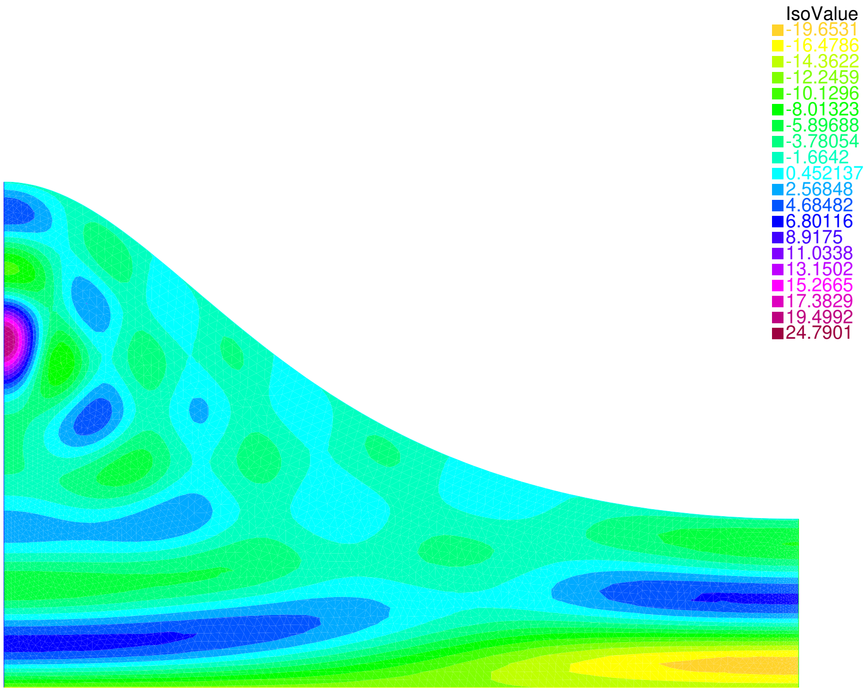

Figure 15: Response plot for $k = 15$

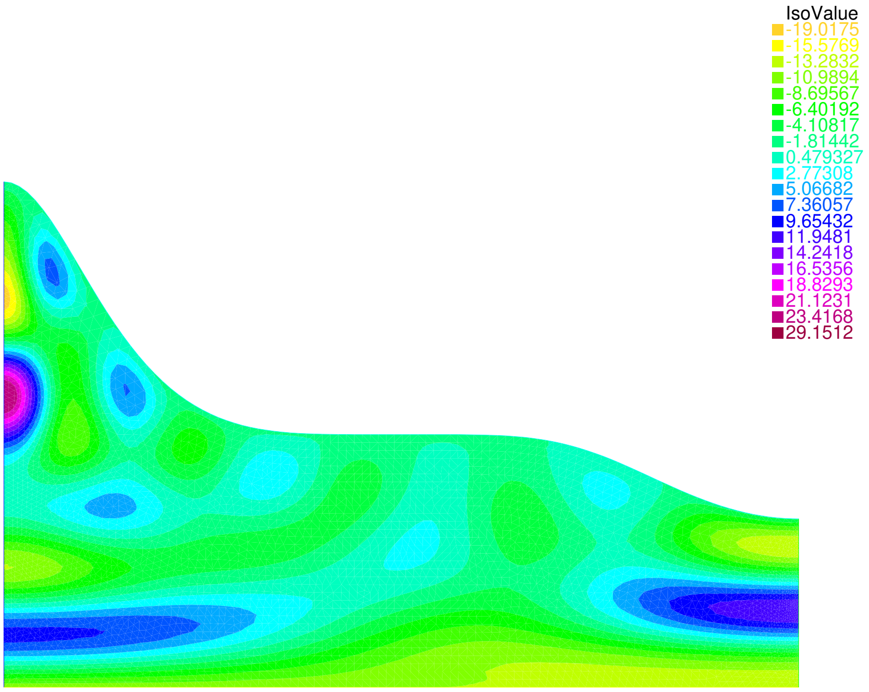

Figure 16: Response plot for $k = 14$ for an oval other than ellipsoid

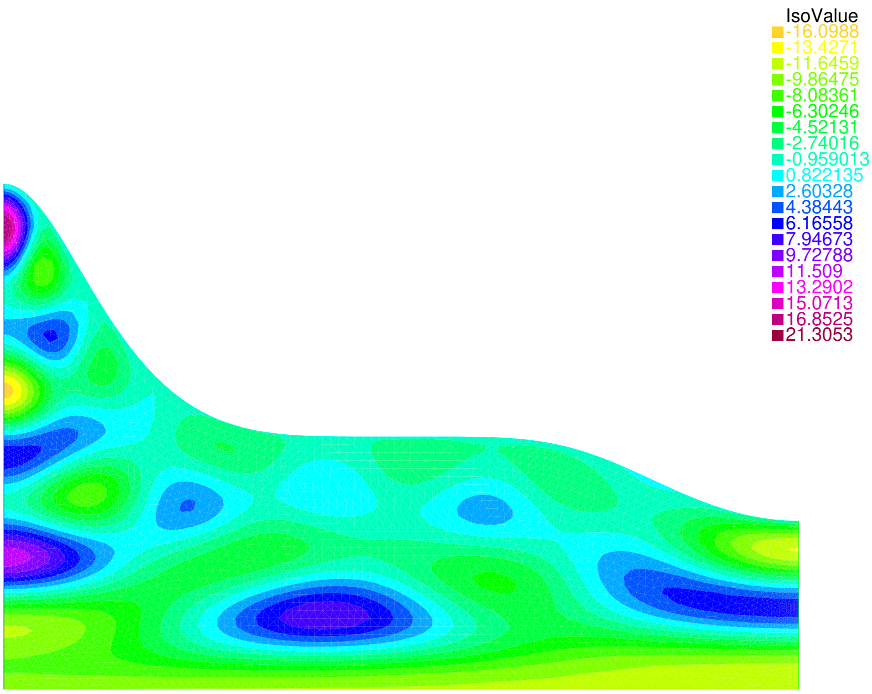

Figure 17: Response plot at $k = 15$ for an oval other than an ellipsoid

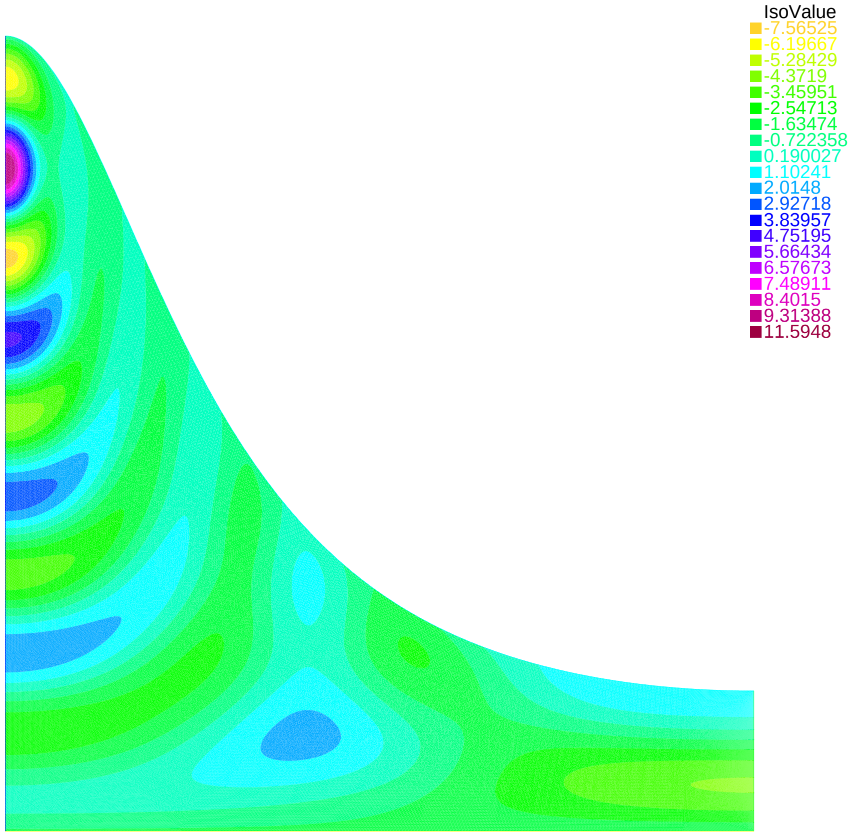

Figure 18: Response plot at $k = 10$ for an ellipsoid with $e = 0.7$

5 Conclusion

The response to a spherical wave emitted from one focus of an ellipsoid of revolution, for sufficiently large k, is a smooth function with a clearly defined extremum at the second focus of the ellipsoid. This extremum is not a singularity of the solution in the full sense of the word, but it can be assumed to represent a numerical singularity. This means that certain series converge poorly in the vicinity of the focus. We observed this phenomenon in the case of the Luneburg lens [6]. It is worth noting that the finite element method copes well with this computational singularity.

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