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Spinor Field in Cosmology with Lyra's Geometry

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Abstract. We have studied the evolution of space-time with nonlinear spinor field in the framework of Lyra's geometry. The role of a nonlinear spinor field in the evolution of Universe was identified. Earlier we have considered the nonlinear spinor field in isotropic and anisotropic cosmological models and found that the presence of nontrivial non-diagonal terms in energy-momentum tensor imposes different type of restrictions both on space-time geometry and spinor field itself. The introduction of Lyra's geometry leads to the complex dependence of invariants of bilinear spinor forms.

Keywords: spinor field; cosmology; Lyra's Geometry

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1. Introduction

The Standard Model of Cosmology (SMC), also known as the Λ-CDM model (Λ-Cold Dark Matter), rests on three fundamental assumptions:

(a) the validity of General Relativity on cosmological scales;

(b) the correctness of the Standard Model of particle physics at small (quantum) scales; and

(c) the cosmological principle, which posits that the Universe is spatially homogeneous, isotropic, and infinite on large scales.

According to this model, the Universe originated from a Big Bang, emerging from a state of pure energy. The present-day energy composition of the Universe is estimated to be approximately `5\%` ordinary (baryonic) matter, `27\%` dark matter, and `68\%` dark energy.

Despite its simplicity, the Λ-CDM model successfully explains a wide range of cosmological observations, including Type Ia supernovae, cosmic microwave background radiation (CMBR) anisotropies, large-scale structure formation, gravitational lensing, and baryon acoustic oscillations. However, it faces theoretical challenges, notably severe fine-tuning problems related to the vacuum energy (cosmological constant) scale. These shortcomings motivate the exploration of alternative cosmological models. In such alternatives, researchers often seek to modify Einstein's field equations by introducing additional terms in the gravitational Lagrangian beyond the Ricci scalar or by considering non-Riemannian geometries. Some approaches also involve exotic matter or field sources.

Shortly after Einstein proposed his famous theory of gravity, Weyl in an attempt to unify gravitation and electromagnetic field, introduced a generalization of Riemannian Geometry [1]. Weyl theory was not taken seriously as it contradicted some well-known observational result. In 1951 Lyra proposed a modification of Riemannian geometry which bears a close resemblance of Weyl geometry [2]. But unlike Weyl geometry, in Lyra's geometry the connection is metric preserving as in Riemannian geometry. In doing so he introduced a gauge function into the structureless manifold. This theory was further developed by Sen [3], Halford [4], Sen and Dunn [5], Sen and Vanstone [6] and many others. Recently Lyra's geometry is being used extensively in cosmology [7, 8, 9, 10, 11].

In a number of papers [12, 13, 14] it was shown that spinor field is very sensitive to the gravitational one. In most cases there exist nontrivial non-diagonal components of energy-momentum tensor (EMT) which leads to the different types of restrictions both on the geometry of space-time and the spinor field itself. The aim for considering Lyra's geometry is to clarify whether it can remove or weaken the restrictions those occur in usual cases.

2. Basic equations

2.1 Riemann geometry

An affine connection is characterized by its components $\Gamma^{\mu}_{\alpha\beta}$ which are defined by the change due to infinitesimal parallel transform of a vector ξ^{μ} from a point $P(X^{\mu})$ to a point $P(x^{\mu}+dx^{\mu})$:

$$\delta\xi^{\mu} = -\Gamma^{\mu}_{\alpha\beta}\xi^{\alpha}dx^{\beta}, \tag{1}$$

and the fundamental metric tensor $g_{\mu\nu}$ that is defined the measure of length ξ of a vector ξ^{μ}:

$$\xi^2 = \xi_{\mu}\xi^{\mu} = g_{\mu\nu}\xi^{\mu}\xi^{\nu}. \tag{2}$$

From the foregoing identity we obtain

$$2\xi\delta\xi = g_{\mu\nu,\alpha}\xi^{\mu}\xi^{\nu}dx^{\alpha} + g_{\mu\nu}\delta\xi^{\mu}\xi^{\nu} + g_{\mu\nu}\xi^{\mu}\delta\xi^{\nu}, \tag{3}$$

where $g_{\mu\nu,\alpha} = \partial g_{\mu\nu}/\partial x^{\alpha}$. Inserting (1) into (3) after some manipulation we find

$$2\xi\delta\xi = g_{\mu\nu;\alpha}\xi^{\mu}\xi^{\nu}dx^{\alpha}, \tag{4}$$

where

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^{\beta}_{\mu\alpha}\,g_{\beta\nu} - \Gamma^{\beta}_{\nu\alpha}\,g_{\mu\beta} \tag{5}$$

is the covariant derivative of metric function. In Riemann geometry $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ and under parallel transform the length does not change. This leads to the metricity condition

$$g_{\mu\nu;\alpha} = 0. \tag{6}$$

Note that from (6) one finds the connection $\Gamma^{\alpha}_{\mu\nu}$ which we further denote as $\{^{\alpha}_{\mu\nu}\}$, is symmetric in two lower indices and known as Levi-Civita connection. Summation of (6) and its cyclic cunterparts

$$g_{\mu\nu;\alpha} + g_{\nu\alpha;\mu} + g_{\alpha\mu;\nu} = 0 \tag{7}$$

yields

$$\{^{\alpha}_{\mu\nu}\} = \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right). \tag{8}$$

2.2 Weyl's geometry:

In electrodynamics it was found that Maxwell's equations are invariant to a certain change in electromagnetic 4-potential $\tilde{A}_\mu = A_\mu + \partial_\mu f$ with f being a scalar. Under this the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ remains unchanged. It is known as gauuge in electrodynamics. In 1918 Weyl proposed a new kind of gauge theory involving metric tensor $g_{\mu\nu}$. According to his assumption there exist a geodesic gauge in which lenth of a vector does not change under parallel transform, but in an arbitray gauge it is assumed to change. Thus in analogy with (2) he proposed

$$d\xi = -\xi \phi_\mu dx^\mu, \tag{9}$$

where ϕ_μ is a vector function characterizing the manifold. Thus the metrical connection of a Weyl manifold is characterized by two independent quantities $g_{\mu\nu}$ and ϕ_μ. In this case length is no longer remains unchanged. In fact inserting (9) into (3) we find

$$-2\phi_\alpha g_{\mu\nu} = g_{\mu\nu,\alpha} - \Gamma^\beta_{\mu\alpha}\, g_{\beta\nu} - \Gamma^\beta_{\nu\alpha}\, g_{\mu\beta}, \tag{10a}$$

$$-2\phi_\mu g_{\nu\alpha} = g_{\nu\alpha,\mu} - \Gamma^\beta_{\nu\mu}\, g_{\beta\alpha} - \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu}, \tag{10b}$$

$$-2\phi_\nu g_{\alpha\mu} = g_{\alpha\mu,\nu} - \Gamma^\beta_{\alpha\nu}\, g_{\beta\mu} - \Gamma^\beta_{\mu\nu}\, g_{\beta\alpha}, \tag{10c}$$

from which after a little manipulation we find

$$\bar{\Gamma}^\alpha_{\mu\nu} = \{^{\,\alpha}_{\mu\nu}\} + \frac{1}{2}\left(\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu}\phi^\alpha\right), \quad \phi^\mu = g^{\mu\nu}\phi_\nu. \tag{11}$$

If one makes a gauge transformation $\xi \to \bar{\xi} = \lambda\xi, \quad \lambda = \lambda(x)$, then

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}, \quad \phi_\mu = \phi_\mu - 2\lambda_\mu/\lambda, \quad \lambda_\mu = \partial\lambda/\partial x^\mu \tag{12}$$

leaves (11) unaltered.

2.3 Lyra's geometry:

Lyra suggested a modification of Riemannian geometry which is also a modification of Weyl geometry. The metrical concept of gauge in Weyl geometry was modified by a structureless gauge function. The displacement vector between two neighbouring points now has the components $\xi^\mu = x^0 dx^\mu$ where x^0 and is a nonzero gauge function. The transformation to new reference system is given by

$$x^\mu = x^\mu(x^{\lambda'}), \quad x^0 = x^0(x^{\mu'}, x^{0'}), \tag{13}$$

where $\partial x^0/\partial x^{0'}$ and Det $\partial x^\mu/\partial x^{\nu'} \neq 0$. Under this transformation components of a contravariant vector are transformed according to

$$\xi^\mu = \lambda^{-1}\frac{\partial x^\mu}{\partial x^{\nu'}}\xi^{\nu'}, \quad \lambda = \frac{x^{0'}}{x^0}. \tag{14}$$

From the definition of affine connecxion we know that there exists for everypoint P in a local reference system $(x^{0'}, x^{\mu'})$ in the immediate neighbourhood, known as geodesic at P, such that it is $\delta\xi^{\mu'} = 0$. Then in any general reference system (x^0, x^μ) we have

$$\delta\xi^\mu = \left(\frac{1}{x^0}\frac{\partial^2 x^\mu}{\partial x^{\beta'}\partial x^{\eta'}}\frac{\partial x^{\beta'}}{\partial x^\alpha}\frac{\partial x^{\nu'}}{\partial x^\eta} - \frac{1}{2}\delta^\mu_\eta\phi_\alpha\right)\xi^\eta x^0 dx^\alpha, \quad \phi_\alpha = \frac{1}{x^0}\frac{\partial \ln\lambda^2}{\partial x^\alpha}. \tag{15}$$

Therefore, in any general reference system (x^0, x^α) the parallel transfer of a vector from $P = (x^\mu)$ to $P' = (x^\mu + dx^\mu)$ can be written as

$$\delta\xi^\mu = -\tilde{\Gamma}^\mu_{\alpha\beta}\xi^\alpha x^0 dx^\beta = -\left(\Gamma^\mu_{\alpha\beta} - \frac{1}{2}\delta^\mu_\alpha\phi_\beta\right)\xi^\alpha x^0 dx^\beta. \tag{16}$$

The interval in Lyra's geometry is given by

$$ds^2 = g_{\mu\nu}x^0 dx^\mu x^0 dx^\nu. \tag{17}$$

The parallel transport of length in Lyra geometry is integrabble, i.e.,

$$\delta(g_{\mu\nu}\xi^\mu\xi^\nu) = \frac{1}{x^0}\partial_\alpha g_{\mu\nu}\xi^\mu\xi^\nu x^0 dx^\alpha + g_{\mu\nu}\delta\xi^\mu\xi^\nu + g_{\mu\nu}\xi^\mu\delta\xi^\nu = 0. \tag{18}$$

Inserting (15) into (18) we obtain

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{x^0}\{^{\alpha}_{\mu\nu}\} + \frac{1}{2}\left(\delta^\alpha_\mu\phi_\nu + \delta^\alpha_\nu\phi_\mu - g_{\mu\nu}\phi^\alpha\right), \tag{19}$$

which is similar to the connection (11) of Weyl geometry. Here $\{^{\alpha}_{\mu\nu}\}$ is the Levi-Civita connection. It should be noted that $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$, though $\tilde{\Gamma}^\mu_{\alpha\beta} \neq \tilde{\Gamma}^\mu_{\beta\alpha}$. Moreover, $\Gamma^\mu_{\alpha\beta}$ in Weyl's and Lyra's geometries differs from each other by the factor $1/x^0$. In Lyra's geometry we should substitute $\Gamma^\rho_{\mu\nu}$ with $\tilde{\Gamma}^\rho_{\mu\nu}$. On account of (19) we can now rewrite $\tilde{\Gamma}^\rho_{\mu\nu}$ as

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{1}{x^0}\{^{\mu}_{\alpha\beta}\} + \frac{1}{2}\left(\delta^\mu_\beta\phi_\alpha - g_{\alpha\beta}\phi^\mu\right). \tag{20}$$

The parallel transfer, hence the equation of motion

$$\frac{1}{x^0}\frac{\partial\xi^\alpha}{\partial\beta} + \tilde{\Gamma}^\alpha_{\nu\beta}\xi^\nu = 0 \tag{21}$$

is integrable if the components of the tensor

$$K^{\lambda}_{\mu\alpha\beta} = \frac{1}{(x^0)^2}\Big[\frac{\partial(x^0\tilde{\Gamma}^{\lambda}_{\mu\beta})}{\partial x^{\alpha}} - \frac{\partial(x^0\tilde{\Gamma}^{\lambda}_{\mu\alpha})}{\partial x^{\beta}} + x^0\tilde{\Gamma}^{\lambda}_{\rho\alpha}x^0\tilde{\Gamma}^{\rho}_{\mu\beta} - x^0\tilde{\Gamma}^{\lambda}_{\rho\beta}x^0\tilde{\Gamma}^{\rho}_{\mu\alpha}\Big] \tag{22}$$

vanish. As one sees, it is the analog Riemann tensor in Lyra's geometry and can be expressed

$$K^{\lambda}_{\mu\alpha\beta} = \star R^{\lambda}_{\mu\alpha\beta} + \frac{1}{2}\delta^{\lambda}_{\mu}\Phi_{\alpha\beta} \tag{23}$$

with

$$\star R^{\lambda}_{\mu\alpha\beta} = \frac{1}{x^0}\left[\frac{\partial\Gamma^{\lambda}_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{\lambda}_{\mu\alpha}}{\partial x^{\beta}}\right] + \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\rho}_{\mu\beta} - \Gamma^{\lambda}_{\rho\beta}\Gamma^{\rho}_{\mu\alpha} + \frac{1}{2}\left(\mathring{\phi}_{\alpha}\Gamma^{\lambda}_{\mu\beta} - \mathring{\phi}_{\beta}\Gamma^{\lambda}_{\mu\alpha}\right), \tag{24}$$

$$\Phi_{\alpha\beta} = \frac{1}{x^0}\left[\frac{\partial\phi_{\alpha}}{\partial x^{\beta}} - \frac{\partial\phi_{\beta}}{\partial x^{\alpha}}\right] + \frac{1}{2}\left(\mathring{\phi}_{\alpha}\phi_{\beta} - \mathring{\phi}_{\beta}\phi_{\alpha}\right). \tag{25}$$

As was shown by Sen in a normal gauge with $x^0 = 1$ Einstein's feild equations in Lyra's geometry take the form

$$G^{\nu}_{\mu} + \frac{3}{2}\phi_{\mu}\phi^{\nu} - \frac{3}{4}\delta^{\nu}_{\mu}\phi_{\alpha}\phi^{\alpha} = \kappa T^{\nu}_{\mu}, \tag{26}$$

where ϕ_{μ} is the displacement vector. Here

$$G^{\nu}_{\mu} = R^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}R. \tag{27}$$

Let us consider ϕ_{μ} as a time-like vector field of displacement.

2.4 Spinor field

Given the role that spinor field can play in the evolution of the Universe, question that naturally pops up is, if the spinor field can redraw the picture of evolution caused by perfect fluid and dark energy, is it possible to simulate perfect fluid and dark energy by means of a spinor field? Affirmative answer to this question was given in the a number of papers. We consider the spinor field Lagrangian given by [12]

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right] - m_{\rm sp}\bar{\psi}\psi - F, \tag{28}$$

where the nonlinear term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants K that take one of the following values $\{I,\ J,\ I+J,\ I-J\}$ generated from the real bilinear forms of a spinor field. We also consider the case $\psi = \psi(t)$, so that $I = S^2 = (\bar{\psi}\psi)^2$, & $J = P^2 = (\imath\bar{\psi}\gamma^5\psi)^2$.

The spinor field equations take the form

$$\imath\gamma^\mu\nabla_\mu\psi - m_{\rm sp}\psi - \mathcal{D}\psi - \imath\mathcal{G}\gamma^5\psi = 0, \tag{29}$$
$$\imath\nabla_\mu\bar\psi\gamma^\mu + m_{\rm sp}\bar\psi + \mathcal{D}\bar\psi + \imath\mathcal{G}\bar\psi\gamma^5 = 0, \tag{30}$$

where we denote $\mathcal{D} = 2SF_K K_I$ and $\mathcal{G} = 2PF_K K_J$ with $F_K = dF/dK$, $K_I = dK/dI$ and $K_J = dK/dJ$. In the Lagrangian (28) and spinor field equations (29) and (30), ∇_μ is the covariant covariant derivative of the spinor field, so that $\nabla_\mu\psi = \partial_\mu - \Omega_\mu\psi$ and $\nabla_\mu\bar\psi = \partial\bar\psi + \bar\psi\Omega_\mu$, where

$$\Omega_\mu = \frac{1}{4}\bar\gamma_a\gamma^\nu\partial_\mu e_\nu^{(a)} - \frac{1}{4}\gamma_\rho\gamma^\nu\Gamma^\rho_{\mu\nu}. \tag{31}$$

As far as Lyra geometry is concerned, in (31) we should replace $\Gamma^\rho_{\mu\nu}$ by $\tilde\Gamma^\rho_{\mu\nu}$. Taking into account that in Lyra's geometry ∂/∂_μ is replaced by $\partial/(x^0\partial_\mu)$. In view of (20) we rewrite $\tilde\Gamma^\rho_{\mu\nu}$ as

$$\tilde\Gamma^\rho_{\mu\nu} = \frac{1}{x^0}\{^{\,\rho}_{\mu\nu}\} + \frac{1}{2}\left(\delta^\rho_\nu\phi_\mu - g_{\mu\nu}\phi^\rho\right). \tag{32}$$

Inserting it into (31) one finds the spinor affine connection

$$\begin{aligned}\tilde\Omega_\mu &= \frac{1}{x^0}\left[\frac{1}{4}\bar\gamma_a\gamma^\nu\partial_\mu e_\nu^{(a)} - \frac{1}{4}\gamma_\rho\gamma^\nu\{^{\,\rho}_{\mu\nu}\}\right] - \frac{1}{8}\left(\gamma_\nu\gamma^\nu\phi_\mu + \gamma_\mu\gamma^\nu\phi_\nu\right)\\ &= \frac{1}{x^0}\mathring\Omega_\mu + \bar\Omega_\mu.\end{aligned} \tag{33}$$

The energy momentum tensor of the spinor field is given by

$$\begin{aligned}T_\mu{}^\rho &= \frac{\imath g^{\rho\nu}}{4}\big(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\big) - \delta^\rho_\mu L_{\rm sp}\\ &= \frac{\imath}{4}g^{\rho\nu}\big(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi\big)\\ &- \frac{\imath}{4}g^{\rho\nu}\bar\psi\big(\gamma_\mu\tilde\Omega_\nu + \tilde\Omega_\nu\gamma_\mu + \gamma_\nu\tilde\Omega_\mu + \tilde\Omega_\mu\gamma_\nu\big)\psi\\ &- \delta^\rho_\mu\big(2KF_K - F(K)\big).\end{aligned} \tag{34}$$

On account of spinor field equations (29) and (30) the spinor field Lagrangian takes the form $L_{\rm sp} = 2KF_K - F(K)$. The term in red is responsible for non-diagonal components. Thanks to spinor field equations the conservation of energy holds, i.e.,

$$T^\mu_{\nu;\mu} = 0. \tag{35}$$

Then taking into account that the in case of spinor field $T^\nu_{\mu;\nu} = 0$ on accout on Bianchi identity $G^\nu_{\mu;\nu} = 0$ from (26) we find

$$\left(\frac{3}{2}\phi_\mu\phi^\nu - \frac{3}{4}\delta^\nu_\mu\phi_\alpha\phi^\alpha\right)_{;\nu} = 0. \tag{36}$$

Following Sen we consider the gauge function as follows:

$$\phi_\mu = \{\beta(t),\, 0,\, 0,\, 0\}. \tag{37}$$

For Bianchi metrics in this case we find the additional part of spinor affine connection $\bar{\Omega}_\mu \propto \beta$. Unlike Riemann geometry in this case the invariants of spinor field possess following form

$$K = \frac{C_0^2}{V^2}\exp\left[-\frac{3\beta_0}{2}\int\frac{dt}{V(t)}\right]. \tag{38}$$

with V being the volume scale of space-time. Now one has to give concrete form of space-time to find the solution to the corresponding Einstein and spinor field equations.

3. Conclusion

Within the scope of gravitational cosmological models with Lyra's geometry we have studied the role of nonlinear spinor field in the evolution of universe. Earlier studies showed that the presence of nontrivial non-diagonal terms in energy-momentum tensor imposes different type of restrictions both on space-time geometry and spinor field itself. The introduction of Lyra's geometry leads to the complex dependence of invariants of bilinear spinor forms. In this report we give some general ideas about spinor field with Lyra's geometry. We plan to consider some specific cases in future.

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