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Gauss's lemma and integration of a linear ordinary differential equation

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Abstract. The problem of factorization of second-order linear ordinary differential equations (LODEs) is considered. Over a polynomial ring $k[x]$, the operator $L = aD^2 + bD + c$ is considered, where D is the differentiation operator with respect to x, and the coefficients a, b, c are polynomials from the ring $k[x]$. A Gaussian integral factor is introduced as a polynomial μ such that μL can be factorized into linear factors with coefficients from the ring $k[x]$. If the operator L can be factorized over the field of quotients of the ring $k[x]$, then a Gaussian factor exists. However, in contrast to the classical case of factorization in the ring $\mathbb{Q}[x]$, it is impossible to prove a complete analogue of the Gauss lemma, that is, to assume that $\mu = 1$. A possible generalization of Gauss's lemma to this case is presented, and from it a connection is derived between the Gaussian multiplier and finite-zero solutions of linear ordinary differential equations (LODEs).

Keywords: linear ordinary differential equations, symbolic integration, integrating multiplier

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1. Introduction

The question of integrating finite-state linear ordinary differential equations (LODEs) has attracted interest for several centuries. Currently, this issue has been relegated to computer algebra, and several recent publications on the topic have appeared [1].

Historically, the first method to appear was based on the analogy between linear differential equations and algebraic equations, systematically presented by Libri in the 1830s [2]. This approach was widely known to contemporaries; indeed, a Russian translation of Brassina's memoir on this subject was included as an appendix in the Russian translation of Sturm's Course in Analysis. At the end of the 20th century, this approach was developed and translated into the language of computer algebra by L.M. Berkovich under the name "factorization method" [3, 4, 5].

Consider a second-order linear ODE

$$ay'' + by' + cy = 0 \tag{1}$$

or in operator form

$$(aD^2 + bD + c)y = 0,$$

where the coefficients a, b, c belong to some integral ring R with differentiation D. The factorization method consists of studying the decomposition of the differential operator $L = aD^2 + bD + c$ into linear factors, i.e., finding four functions $\alpha_i, \beta_i \in R$ such that

$$L = aD^2 + bD + c = (\alpha_1 D + \beta_1)(\alpha_2 D + \beta_2).$$

The factorization coefficients $\alpha_1, \dots, \beta_2$ are usually sought in the field of fractions of the polynomial ring $\mathbb{C}[x]$ or its quadratic extensions. Repeated attempts to consider more general extensions by analogy with Galois theory do not lead to new cases of factorization [5, note 6 to Chapter 1]. A broad class of operators that admit factorization into is given by the main theorem of [4].

Besides the factorization method, a seemingly completely different method for solving ODEs, the integrating factor method, is often discussed nowadays. This method underlies most symbolic methods for integrating first-order differential equations [6, 7, 8]. Since Cheb-Terrab's memoir [9] on second-order equations, the following generalization of the integrating factor has come into use. $\mu(x, y, y')$ is said to be an integrating factor of an ODE $f(x, y, y, y', y'') = 0$ if there exists a function $g(x, y, y')$ such that

$$f = \mu \frac{dg}{dx}.$$

In this case,

$$g(x, y, y') = \text{Const}$$

is the first integral of the original second-order ODE.

When integrating a second-order ODE, it is natural to want to obtain a first-order ODE as a first integral; this kind of construction is considered, e.g., in [1].

In this case, it is natural to assume that μ depends only on x, and $g(x, y, y') = \alpha(x)y' + \beta(x)y$. But then

$$f(x, y, y', y'') = ay'' + by' + cy = \mu \frac{d}{dx}(\alpha y' + \beta)$$

or

$$L = \mu D(\alpha D + \beta).$$

Thus, the integrating factor method is a particular case of the factorization method.

However, the integrating factor method appears to be a special case of the factorization method only as long as we consider factorization over a field. It is quite natural to attempt factorization by analogy with the factorization of a polynomial with integer coefficients. This analogy is what we intend to consider in this article.

2. Gaussian factor

Recall that in computer algebra, factorization of a polynomial with integer coefficients is performed over the ring $\mathbb{Z}$, not over the field $\mathbb{Q}$. This does not create any problems, since Gauss's lemma [10] can be proved: if a polynomial f is factorizable over $\mathbb{Q}$, that is, if there exists an integer μ such that

$$\mu f = g \cdot h, \quad g, h \in \mathbb{Z}[x],$$

then there exist polynomials $g', h' \in \mathbb{Z}[x]$ such that

$$f = g' \cdot h'.$$

In complete analogy with what has been said, assume that the coefficients a, b, c of the operator L belong to the polynomial ring $\mathbb{C}[x]$, and the operator itself admits factorization over the field of fractions of this ring:

$$L = (D + \gamma_1)(D + \gamma_2).$$

Representing γ_1, γ_2 as a ratio of polynomials, we rewrite this expression as

$$\mu L = (\alpha_1 D + \beta_1)(\alpha_2 D + \beta_2), \tag{2}$$

where $\alpha_1, \ldots, \mu \in \mathbb{C}[x]$.

If the operator L factorizes over the field of fractions of the ring $k[x]$, then a Gaussian factor exists. By analogy with the case of ordinary polynomials, it is tempting to assume that $\mu = 1$ without loss of generality. However, we have no reason to believe that such a generalization of Gauss's lemma is valid. A significantly weaker assertion can be proved.

Theorem 1. *Let k be one of the standard number fields $\mathbb{Q}, \mathbb{A}, \mathbb{R}, \mathbb{C}$. If the operator L with coefficients in $k[x]$ can be factorized over the field of fractions of the ring $k[x]$, then it can be represented as*

$$\mu L = (\alpha_1 D + \beta_1)(\mu\alpha_2 D - \alpha_2\mu' + \mu\beta_2), \tag{3}$$

where $\alpha_1, \ldots, \mu \in k[x]$ and the factor μ has no common factors with α_1.

Proof. Factorization of (2) means that

$$\begin{cases} \alpha_1\alpha_2 = \mu a, \\ \alpha_1\alpha_2' + \alpha_1\beta_2 + \alpha_2\beta_1 = \mu b, \\ \alpha_1\beta_2' + \beta_1\beta_2 = \mu c. \end{cases} \tag{4}$$

Let p be a prime factor of μ. If α_2 is not divisible by p, then by virtue of the first equation, α_1, is divisible by p, and by virtue of the second, β_1 is divisible by p. Therefore, in this case, the left-hand factor can be reduced by p. This allows, without loss of generality, to retain in μ only the divisors of α_2:

$$\alpha_2 = \mu\overline{\alpha}_2, \quad \overline{\alpha}_2 \in k[x].$$

But then

$$\begin{cases} \alpha_1\overline{\alpha}_2 = a, \\ \alpha_1(\overline{\alpha}_2\mu' + \beta_2) + \mu(\alpha_1\overline{\alpha}_2' + \overline{\alpha}_2\beta_1) = \mu b, \\ \alpha_1\beta_2' + \beta_1\beta_2 = \mu c. \end{cases}$$

By assumption, α_1 has no common divisors with μ, so the second equation implies that

$$\overline{\alpha}_2\mu' + \beta_2 = \mu\overline{\beta}_2, \quad \overline{\beta}_2 \in k[x].$$

Substituting the resulting expressions for α_2, β_2 into (2), we obtain (3). □

Remark 1. *It's worth noting that the second equation of the system (4) can be reduced by μ and written*

$$\alpha_1\overline{\beta}_2 + \alpha_1\overline{\alpha}_2' + \overline{\alpha}_2\beta_1 = b,$$

and the third equation can be reduced to the form

$$\alpha_1(\mu\overline{\beta}_2 - \mu'\overline{\alpha}_2)' + \beta_1(\mu\overline{\beta}_2 - \mu'\overline{\alpha}_2) = \mu c$$

or

$$(\alpha_1\overline{\beta}_2' + \beta_1\overline{\beta}_2 - c)\mu + (\alpha_1\overline{\beta}_2 - \alpha_1\overline{\alpha}_2' - \beta_1\overline{\alpha}_2)\mu' - a\mu'' = 0$$

or, taking into account the second equation,

$$(\alpha_1\overline{\beta}_2' + \beta_1\overline{\beta}_2 - c)\mu + (2\alpha_1\overline{\beta}_2 - b)\mu' - a\mu'' = 0.$$

Thus, the system of equations for the coefficients takes the form

$$\begin{cases} \alpha_1\overline{\alpha}_2 = a, \\ \alpha_1\overline{\beta}_2 + \alpha_1\overline{\alpha}_2' + \overline{\alpha}_2\beta_1 = b, \\ (\alpha_1\overline{\beta}_2' + \beta_1\overline{\beta}_2 - c)\mu + (2\alpha_1\overline{\beta}_2 - b)\mu' - a\mu'' = 0. \end{cases} \tag{5}$$

However, the last equation cannot be reduced by μ.

We propose calling μ the Gaussian multiplier of the LDE.

Definition 1. *Let the coefficients of the operator L belong to the polynomial ring $k[x]$. We say that L admits a Gaussian integral factor μ in this ring if μL can be factored into linear factors with coefficients in the ring $k[x]$.*

3. Gaussian multiplier and solution zeros

A solution to the LODE usually has infinitely many zeros in the complex plane. The very distinctive form of the second factor in the expansion (3) allows the existence of a factor with a solution having a finite number of zeros.

Theorem 2. *If the operator L with coefficients in the polynomial ring $\mathbb{C}[x]$ admits a Gaussian integrating factor μ, then all zeros of the factor other than the zeros of the polynomial a are simple. Moreover, the LODE $L(y) = 0$ has a nontrivial solution that vanishes where and only where the factor μ vanishes.*

Proof. Multiplying the equation $L[y] = 0$ by the factor μ does not add new solutions, so the equation

$$\mu\alpha_2 y' - \alpha_2\mu' y + \mu\beta_2 y = 0$$

gives one of the solutions to $L[y] = 0$, namely

$$y_1 = \exp\left(\int \frac{\mu' dx}{\mu} - \int \frac{\beta_2}{\alpha_2} dx\right) = \mu \cdot \exp\left(-\int \frac{\beta_2}{\alpha_2} dx\right).$$

Let us now recall that μ, α_2, β_2 are polynomials, so the solution written out has singularities only at those points where α_2 vanishes. The equality

$$\alpha_1\alpha_2 = a,$$

already used in the proof of Theorem 1 means that the zeros of α_2 are the zeros of the polynomial a. At these points, the exponential can have a zero of any order.

Outside singular points, the exponential does not vanish, so this solution vanishes outside singular points where and only where μ vanishes. From this it is clear that y_1 has zeros only where μ or a vanish. This asserts the second part of the theorem. To prove the first, assume that μ has a multiple zero different from the singular point. Then the constructed solution also has a multiple zero. By Cauchy's theorem, this means that the constructed solution is trivial, which is incorrect.

□

The proven theorem suggests how factorization can be structured.

Example 1. *For example, almost all solutions of the equation*

$$y'' + y = 0$$

have infinitely many zeros. But there is also an exception – the solution e^{ix}, which has no zeros. The coefficient of y'' is 1, so there are no singularities. This means that the Gaussian multiplier μ is a polynomial of degree 1, which is in complete agreement with the expansion

$$D^2 + 1 = (D + i)(D - i).$$

Example 2. *The Bessel equation*

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

for $n = 1/2$ has solutions

$$x^{-1/2} e^{\pm ix},$$

which vanish nowhere and have a singularity where the coefficient of y'' vanishes. All other solutions have infinitely many zeros. Therefore, in this case too, both the multiplier equals 1 and

$$x^2 D^2 + xD + x^2 - \frac{1}{4} = \left(xD + ix - \frac{1}{2}\right)\left(xD + ix + \frac{1}{2}\right).$$

The entire monograph [5] contains only one meaningful example, where factorization requires a Gaussian factor of degree 1.

4. Conclusion

An attempt to shift from considering the factorization of LDEs over fields to factorization over polynomial rings led us to Theorem 1, which is in some sense an analogue of Gauss's theorem for factorization in $\mathbb{Q}[x]$, but is significantly more complex. We cannot assert that the factor can always be taken equal to a constant.

This explains why calculating the integrating factor in this theory leads to nontrivial results. The "linking" that occurs in the second factor of the expansion (3) leads to the fact that the factor cannot be canceled, but on the other hand, it is precisely this that allows us to make some statements about the behavior of the factor on the complex plane, summarized in Theorem 2.

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