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Thin-film waveguide Lüneburg lens in the model of adiabatic guided modes

A. A. Egorov[1], **K. P. Lovetsky**[1,a], **A. L. Sevastianov**[2] and **L. A. Sevastianov**[1,3,b]

[1] Peoples' Friendship University of Russia (RUDN), 6, Miklukho-Maklaya St., Moscow, Russian Federation, 117198

[2] Higher School of Economics (HSE), 11, Pokrovsky Bulvar, Moscow, Russian Federation, 109028

[3] Joint Institute for Nuclear Research (JINR), 6, Joliot-Curie St., Dubna, Moscow Region, Russian Federation, 141980

e-mail: [a] sevastianov-la@rudn.ru, [b] lovetskiy-kp@rudn.ru

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Abstract. A mathematical model is presented that describes the processes of propagation and transformation of coherent electromagnetic radiation in a multilayer three-dimensional (3D) smoothly irregular integrated optical waveguide, called the model of adiabatic guided modes. Its presentation and individual applications in smoothly irregular integrated optical waveguides contain two short stories:

- two-dimensional evolution of guided modes is described;
- boundary conditions are formed on non-horizontal planes tangent to media interfaces, which lead to the description of hybridization of guided modes and other interesting phenomena.

The model of adiabatic guided modes generalizes the cross-section method (reference waveguide method) with nonlocal boundary conditions for the transverse guided mode operator in the reference waveguide cross section to the case of two-dimensional evolution, leading to the description of a number of new effects.

Keywords: mathematical model, 2D and 3D geometry, generalized Luneburg lens, vector Maxwell equations, nonlocal boundary conditions, asymptotic method, adiabatic guided modes, quasi-TE and quasi-TM modes, electromagnetic radiation, waveguide optoelectronics, eigenvalues, eigenfunctions, Nelder-Mead simplex method, numerical integration, Cauchy problem, system of ordinary differential equations, system of linear algebraic equations, Tikhonov regularization method
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1 Introduction

The progress of photonics, integrated optics, and waveguide optoelectronics in the last few decades have made it possible to achieve many important scientific results (see, e.g., [1]–[28]). The relevance and significance of studies in these promising areas are associated with both practical applications and prospects for the use of the corresponding high-speed and low-energy devices [8, 9, 15, 18, 19, 22, 23, 24, 25, 26, 27].

In the 70s - 90s of the last century, the most important results in integrated optics were obtained at the Department of Radiophysics, Peoples' Friendship University (UDN), at the Physical Institute, Academy of Sciences of the Soviet Union, at the Institute of Radio Engineering and Electronics, Academy of Sciences of the Soviet Union, at the Mogilev Branch of the Institute of Physics, Academy of Sciences of the Belorussian Soviet Republic, and in a number of other research centers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The active development of the new direction of integrated optics in UDN contributed to the formulation and solution of a number of topical scientific and technical problems [2, 4, 8, 10, 11, 14, 20, 23]. By 1989, one of these urgent and promising problems had been solved. As a result, a system for controlled deposition of thin-film Lüneburg waveguide lenses was developed and assembled [23]. During its test runs, some unsolved problems in the calculation and diagnostics of integrated optical elements with a variable effective refractive index were revealed.

As a result, the researchers faced the task of constructing a mathematical model for the propagation and transformation of coherent electromagnetic radiation in a multilayer three-dimensional (3D) smoothly irregular integrated optical waveguide, which describes the interaction and interference of guided modes supported by such a waveguide structure. In the papers by A.A. Egorov, K.P. Lovetsky, A.L. Sevastianov, and L.A. Sevastianov [20, 23, 26, 28] such a mathematical model was developed. The authors called it the model of adiabatic guided modes.

The adiabatic waveguide propagation of optical radiation in dielectric and fiber-optic waveguides was previously described by the cross-section method in the papers by B.Z. Katsenelenbaum [1], V.V. Shevchenko [3], M.V. Fedoryuk [7], and in integrated optical waveguides by the method of adiabatic guided modes by A.A. Egorov, L.A. Sevastianov and their co-authors [20, 23, 26, 28]. The papers by A.L. Sevastianov [29, 30] substantiated the model of adiabatic guided modes.

2 Setting the mathematical problem of adiabatic guided modes description

Electromagnetic radiation propagating in an integrated optical waveguide satisfies the vector Maxwell equations, which in the SI units have the form (see, e.g., [6, 13, 15]):

$$\operatorname{rot}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{rot}\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

as well as the material equations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \tag{2}$$

where $\varepsilon = \varepsilon_r \varepsilon_0$ is the medium permittivity; $\mu = \mu_r \mu_0$ is the medium permeability; ε_r, μ_r are the relative permittivity and permeability, respectively; ε_0 and μ_0 are the electric and magnetic constant, respectively.

In addition, the tangential boundary conditions at interfaces,

$$\left.\mathbf{H}^\tau\right|_1 = \left.\mathbf{H}^\tau\right|_2, \quad \left.\mathbf{E}^\tau\right|_1 = \left.\mathbf{E}^\tau\right|_2, \tag{3}$$

and the asymptotic boundary conditions at infinity,

$$\left|\mathbf{E}^\tau\right|_{x\to\pm\infty} < +\infty, \quad \left|\mathbf{H}^\tau\right|_{x\to\pm\infty} < +\infty, \tag{4}$$

should be satisfied.

3 The model of adiabatic guided modes in a multilayer waveguide

Let us formulate the restrictions of the class of considered integrated optical (generally, dielectric) multilayer waveguides and electromagnetic radiation propagating through them.

1. The electromagnetic radiation is optical and monochromatic with a given wavelength $\lambda \in [380; 780]$ nm.

2. The thickness of the guiding layer (core) of the basic thin-film waveguide is comparable with the wavelength of the propagating monochromatic electromagnetic radiation, $d \sim \lambda$.

3. The surface of the additional waveguide layer ($x = h(y, z)$) satisfies the following limitations: $\left|\frac{\partial h}{\partial y}\right|, \left|\frac{\partial h}{\partial z}\right| \ll 1$.

4. The integrated optical waveguide is a material medium consisting of dielectric subdomains filling together all the three-dimensional space.

5. The permittivities of the subdomains are different and real-valued, and the permeability equals the magnetic constant (vacuum permeability) everywhere.

6. The number of layers in the waveguide is limited by the computing facilities and the required accuracy of the computations.

7. There are no external currents and charges. It follows that in the absence of foreign currents and charges, the induced currents and charges are zero.

8. We use a Cartesian system of coordinates introduced as follows: the interfaces between dielectric media of the base three-layered waveguide are parallel to the plane yOz. The space subdomains corresponding to coating and substrate layer are infinite, the additional guiding layers are asymptotically parallel to the yOz-plane. Therefore, $\varepsilon = \varepsilon(x)$.

In the Cartesian coordinates, related to the geometry of the substrate (or three-layered planar dielectric waveguide, underlying the smoothly irregular integrated optical waveguide) the Maxwell equations with the above restrictions taken into account have the form:

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{\varepsilon}{c}\frac{\partial E_x}{\partial t}, & \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\mu}{c}\frac{\partial H_x}{\partial t}, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{\varepsilon}{c}\frac{\partial E_y}{\partial t}, & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\mu}{c}\frac{\partial H_y}{\partial t}, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{\varepsilon}{c}\frac{\partial E_z}{\partial t}, & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\mu}{c}\frac{\partial H_z}{\partial t}, \end{aligned} \tag{5}$$

Note that variable x is fast and variables y, z are slow relative to the small (dimensioned) parameter $1/\omega$. The solutions to Maxwell's equations (5) approximate within the asymptotic method are sought in the form taking into account the separation of slow and fast variables:

$$\mathbf{E}(x, y, z, t) = \sum_{s=0}^{\infty} \frac{\mathbf{E}_s(x; y, z)}{(-i\omega)^{\gamma+s}} \exp\{i\omega t - ik_0\varphi(y, z)\}, \tag{6}$$

$$\mathbf{H}(x, y, z, t) = \sum_{s=0}^{\infty} \frac{\mathbf{H}_s(x; y, z)}{(-i\omega)^{\gamma+s}} \exp\{i\omega t - ik_0\varphi(y, z)\}. \tag{7}$$

Retention of zero- and first-order terms with respect to the small parameter in the solution (6), (7) leads to the model of adiabatic guided modes (AGM) describing the guided propagation of a polarized optical radiation through irregular segments of a smoothly irregular (multilayer) optical waveguide. In regular segments, the adiabatic guided modes transform into normal modes of a regular planar optical waveguide.

In the notation $\mathbf{E}_s(x; y, z)$, $\mathbf{H}_s(x; y, z)$ the separation of x by a semicolon denotes the following assumptions:

$$\left\| \frac{\partial \mathbf{E}_s(x; y, z)}{\partial y} \right\|, \left\| \frac{\partial \mathbf{E}_s(x; y, z)}{\partial z} \right\| \sim \frac{1}{\omega} \left\| \frac{\partial \mathbf{E}_s(x; y, z)}{\partial x} \right\|, \; j = x, y, z \tag{8}$$

and

$$\left\| \frac{\partial \mathbf{H}_s(x; y, z)}{\partial y} \right\|, \left\| \frac{\partial \mathbf{H}_s(x; y, z)}{\partial z} \right\| \sim \frac{1}{\omega} \left\| \frac{\partial \mathbf{H}_s(x; y, z)}{\partial x} \right\|, \; j = x, y, z \tag{9}$$

where $\| \, \|$ is the Hilbert norm of functions of x, and ω is the circular frequency of the propagating monochromatic electromagnetic radiation.

3.1 The AWG model equation in the zero-order approximation

In Refs. [20, 23, 28] it is shown that the zero-order approximation (in the asymptotic approach) of the guided-wave solution of the Maxwell equations is given by the following expressions:

$$\begin{Bmatrix} \mathbf{E}(x,y,z,t) \\ \mathbf{H}(x,y,z,t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0(x;y,z) \\ \mathbf{H}_0(x;y,z) \end{Bmatrix} \exp\{i\omega t - i\varphi(y,z)\}, \tag{10}$$

in this case

$$\varepsilon \frac{\partial E_0^y}{\partial x} = -ik_0 \left(\frac{\partial \varphi}{\partial y}\right)\left(\frac{\partial \varphi}{\partial z}\right) H_0^y - ik_0 \left(\varepsilon\mu - \left(\frac{\partial \varphi}{\partial y}\right)^2\right) H_0^z \tag{11}$$

$$\varepsilon \frac{\partial E_0^z}{\partial x} = ik_0 \left(\varepsilon\mu - \left(\frac{\partial \varphi}{\partial z}\right)^2\right) H_0^y + ik_0 \left(\frac{\partial \varphi}{\partial z}\right)\left(\frac{\partial \varphi}{\partial y}\right) H_0^z \tag{12}$$

$$\mu \frac{\partial H_0^y}{\partial x} = ik_0 \left(\frac{\partial \varphi}{\partial y}\right)\left(\frac{\partial \varphi}{\partial z}\right) E_0^y + ik_0 \left(\varepsilon\mu - \left(\frac{\partial \varphi}{\partial y}\right)^2\right) E_0^z \tag{13}$$

$$\mu \frac{\partial H_0^z}{\partial x} = -ik_0 \left(\varepsilon\mu - \left(\frac{\partial \varphi}{\partial z}\right)^2\right) E_0^y - ik_0 \left(\frac{\partial \varphi}{\partial z}\right)\left(\frac{\partial \varphi}{\partial y}\right) E_0^z \tag{14}$$

and

$$E_0^x = -\frac{\partial \varphi}{\partial y}\frac{1}{\varepsilon} H_0^z + \frac{\partial \varphi}{\partial z}\frac{1}{\varepsilon} H_0^y, \tag{15}$$

$$H_0^x = \frac{\partial \varphi}{\partial y}\frac{1}{\mu} E_0^z - \frac{\partial \varphi}{\partial z}\frac{1}{\mu} E_0^y, \tag{16}$$

as well as

$$\left(\frac{\partial \varphi}{\partial y}(y,z)\right)^2 + \left(\frac{\partial \varphi}{\partial z}(y,z)\right)^2 = n_{eff}^2(y,z). \tag{17}$$

For a thin-film multilayer waveguide consisting of optically homogeneous layers, the matching conditions are valid for the electromagnetic field at interfaces between the media,

$$\mathbf{n} \times \mathbf{E}^- + \mathbf{n} \times \mathbf{E}^+ = 0, \tag{18}$$

$$\mathbf{n} \times \mathbf{H}^- + \mathbf{n} \times \mathbf{H}^+ = 0, \tag{19}$$

In addition, the asymptotic conditions hold

$$E_y^0, E_z^0, H_y^0, H_z^0 \xrightarrow[x\to\pm\infty]{} 0. \tag{20}$$

For any fixed (y,z), the system of Eqs. (11),(12),(13), (14),(20) defines a problem of finding eigenvalues $\left(\vec{\nabla}\varphi\right)_j^2(y,z)$ and eigenfunctions $\left(E_y^j, E_z^j, H_y^j, H_z^j\right)^T(y,z)$, normalized to unity:

$$\int_{-\infty}^{+\infty} \left|E_y^j\right|^2 dx = 1, \quad \int_{-\infty}^{+\infty} \left|H_y^j\right|^2 dx = 1. \tag{21}$$

3.2 The AWM model equations in the first-order approximation

We continue using the approach of asymptotic expansion in a small parameter and arrive at a system of equations in the first-order approximation of the method [31]:

$$-\frac{\partial E_1^z}{\partial x}+\frac{ik_0}{\varepsilon}\frac{\partial\varphi}{\partial z}\left(\frac{\partial\varphi}{\partial y}H_1^z-\frac{\partial\varphi}{\partial z}H_1^y\right)+ik_0\mu H_1^y=i\omega\frac{\partial E_0^x}{\partial z}+\frac{i\omega}{\varepsilon}\frac{\partial\varphi}{\partial z}\left(\frac{\partial H_0^y}{\partial z}-\frac{\partial H_0^z}{\partial y}\right)\quad(22)$$

$$\frac{\partial E_1^y}{\partial x}-\frac{ik_0}{\varepsilon}\frac{\partial\varphi}{\partial y}\left(\frac{\partial\varphi}{\partial y}H_1^z-\frac{\partial\varphi}{\partial z}H_1^y\right)+ik_0\mu H_1^z=-i\omega\frac{\partial E_0^x}{\partial y}-\frac{i\omega}{\varepsilon}\frac{\partial\varphi}{\partial y}\left(\frac{\partial H_0^y}{\partial z}-\frac{\partial H_0^z}{\partial y}\right)\quad(23)$$

$$-\frac{\partial H_1^z}{\partial x}+\frac{ik_0}{\mu}\frac{\partial\varphi}{\partial z}\left(\frac{\partial\varphi}{\partial z}E_1^y-\frac{\partial\varphi}{\partial y}E_1^z\right)-ik_0\varepsilon E_1^y=i\omega\frac{\partial H_0^x}{\partial z}-\frac{i\omega}{\mu}\frac{\partial\varphi}{\partial z}\left(\frac{\partial E_0^y}{\partial z}-\frac{\partial E_0^z}{\partial y}\right)\quad(24)$$

$$\frac{\partial H_1^y}{\partial x}-\frac{ik_0}{\mu}\frac{\partial\varphi}{\partial y}\left(\frac{\partial\varphi}{\partial z}E_1^y-\frac{\partial\varphi}{\partial y}E_1^z\right)-ik_0\varepsilon E_1^z=-i\omega\frac{\partial H_0^x}{\partial y}+\frac{i\omega}{\mu}\frac{\partial\varphi}{\partial y}\left(\frac{\partial E_0^y}{\partial z}-\frac{\partial E_0^z}{\partial y}\right)\quad(25)$$

$$E_1^x+\frac{1}{\varepsilon}\left(\frac{\partial\varphi}{\partial y}H_1^z-\frac{\partial\varphi}{\partial z}H_1^y\right)=\frac{1}{\varepsilon}\frac{\omega}{k_0}\left(\frac{\partial H_0^y}{\partial z}-\frac{\partial H_0^z}{\partial y}\right)\quad(26)$$

$$H_1^x+\frac{1}{\mu}\left(\frac{\partial\varphi}{\partial z}E_1^y-\frac{\partial\varphi}{\partial y}E_1^z\right)=-\frac{1}{\mu}\frac{\omega}{k_0}\left(\frac{\partial E_0^y}{\partial z}-\frac{\partial E_0^z}{\partial y}\right)\quad(27)$$

The system of zero-order equations (11),(12), (13),(14),(15),(16) coincides with the system of equations (22),(23), (24),(25),(26),(27), if we nullify the right-hand sides (the contributions with the zero-order quantities).

$$\mathbf{E}(x;y,z)=\mathbf{E}_0(x;y,z)+\frac{i}{\omega}\mathbf{E}_1(x;y,z)\quad(28)$$

$$\mathbf{H}(x;y,z)=\mathbf{H}_0(x;y,z)+\frac{i}{\omega}\mathbf{H}_1(x;y,z)\quad(29)$$

These fields are necessarily complex-valued. Thus, the first-order contributions introduce into the electromagnetic field expressions the characteristic features of leaky modes. The obtained solutions allow analyzing a wide class of smoothly irregular waveguiding structures, including those with a gradient profile of permittivity.

4 Propagation of two adiabatic guided modes through the Lüneburg thin-film generalized waveguide lens

The Lüneburg thin-film generalized waveguide lens (TGWL) with radius R and focal length F is implemented as a thickening (additional waveguide layer) of radius R on a regular planar waveguide. The algorithm of calculating the adiabatic guided mode includes the following stages (see, e.g., [23],[28]).

1. For each guided mode of a regular waveguide, the effective refractive index is calculated by solving the system of equations:

$$\beta(r)/\beta=\exp[\omega(\rho,F)],\quad(30)$$

where $\rho = r\beta(r)/\beta$ and $\omega(\rho, F) = \dfrac{1}{\pi}\int_\rho^1 \dfrac{\arcsin(x/F)}{(x^2-\rho^2)^{1/2}}dx$, using the Nelder-mead deformed polyhedron simplex method, the integral being computed using the Newton-Cotes formulas of the 8-th order.

2. The trajectories of 2D rays are calculated by solving the system of equations $\dfrac{d}{ds}\left(\beta(y,z)\dfrac{dy}{ds}\right) = \dfrac{\partial\beta}{\partial y}(y,z)$, $\dfrac{d}{ds}\left(\beta(y,z)\dfrac{dz}{ds}\right) = \dfrac{\partial\beta}{\partial z}(y,z)$ equivalent to the system

$$\frac{dy}{dz} = V,\ \frac{dV}{dz} = (1+V^2)(B-AV);\ A = \frac{1}{\beta}\frac{\partial\beta}{\partial z},\ B = \frac{1}{\beta}\frac{\partial\beta}{\partial y}. \tag{31}$$

Numerical integration of system (31) with the initial conditions $y_j(z_0) = y_{j0}$, $V_j(z_0) = 0$, is performed by the Runge-Kutta-Fehlberg method of the 6-th order with the accuracy-adapted step choice.

3. The Cauchy problem for system (31) is determined by the initial conditions $y(z_0) = y_0$, $V(z_0) = V_0$. The current value of derivative $V(z)$ specifies the slope of the ray trajectory dy/dz relative to the Oz-axis, so that it also specifies a vector field $(\beta_y, \beta_z)^T$, tangent to the rays determined by Eqs. (31), namely, $\beta_y = \beta\dfrac{dy}{ds}$, $\beta_z = \beta\dfrac{dy}{ds}$, i.e.

$$\beta_y = \beta V(1+V)^{1/2}, \quad \beta_z = \beta V(1+V^2)^{1/2}. \tag{32}$$

4. The system of ordinary differential equations (11)–(16) is solved using the expansion in the system of fundamental solutions, which reduces the problem to a system of linear algebraic equations for the expansion coefficients $(\mathbf{A},\mathbf{B})^T : \hat{M}(\beta)(\mathbf{A},\mathbf{B})^T = 0$.

5. The conditions of solvability of the system of linear algebraic equations is formulated as an algebraic transcendental equation, $\det(\hat{M}(\beta)) = 0$, or a first-order nonlinear partial differential equation

$$F_{Disp}(\beta, \beta_y, \beta_z; h, \partial h/\partial y, \partial h/\partial z; n_s, n_f, n_l, n_c, d) = 0. \tag{33}$$

6. The partial differential equation (33) is solved using the mesh field $(\beta_y, \beta_z)^T(y_k, z_k)$, which finally yields $\{h, \partial h/\partial y, \partial h/\partial z\}$.

7. Using $\{h, \partial h/\partial y, \partial h/\partial z\}$, vertical distributions of the adiabatic guided mode field at the mesh nodes (y_k, z_k) are calculated by Tikhonov regularization method taking into account the energy conservation law and the Pointing vector propagation law.

8. The phase incursion is calculated using Eqs. (10) on the mesh (y_k, z_k) by numerical integration method.

9. The total electromagnetic field of the adiabatic guided mode is calculated at the mesh nodes (y_k, z_k) using Eqs. (10)–(16).

10. Calculations of items 1–9 are repeated for another adiabatic mode.

11. The total field of two (or more) adiabatic guided modes propagating through the Lüneburg TGWL with radius R and focal length F is calculated.

5 Discussion and conclusion

There are at least two very important problems in integrated optics, where it is necessary to take into account the vector nature of the fields. First, when synthesizing various 3D connection elements (lenses, prisms, etc.), it allows implementing high-efficiency energy transfer. Secondly, in an integrated optical RF spectrum analyzer operating in real time, e.g., airborne. The purpose of such a spectrum analyzer is to use an acousto-optic modulator to perform an instantaneous spectral analysis of the incoming radar signal in order to determine, e.g., what is the given aircraft tracked by (another aircraft, a missile, or a ground-based radar station). The advantage of such a spectrum analyzer over an electronic one is that only a few optical elements are required to perform functions that would require hundreds of electronic elements. In this case, optical elements, as a rule, can be integrated in one optical chip.

In the framework of the obtained analytical solution of the vector electrodynamic problem in a smoothly irregular four-layer integrated optical 3D waveguide, the passage of a guided mode (eigenmode) through the Lüneburg TGWL was studied for the first time using numerical simulation. The calculation of the dispersion relation of a four-layer smoothly irregular integrated optical 3D waveguide is carried out in the approximation of the reference waveguide method and in the approximation of the adiabatic mode method, including the shift in the propagation constants of the quasi-TE and quasi-TM modes taken into account. In the zero vector approximation, a full-aperture Lüneburg TGWL was synthesized and the electromagnetic field profile at the focus of such a lens was calculated.

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