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Slowly rotating spacetimes with scalar hair

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Abstract. We consider asymptotically flat, rotating configurations of a self-gravitating real scalar field minimally coupled to Einstein gravity. For the sake of completeness and to introduce convenient notation, we formulate the Einstein-Klein-Gordon equations in the orthonormal tetrad uniquely associated with the metric of a nonstationary axisymmetric spacetime. First, in the stationary case, we study Bianchi identities to find appropriate variants for reducing the Einstein-Klein-Gordon system to a complete subsystem of independent equations. It turns out that this procedure can be performed in three ways. Second, assuming that the rotation is slow, the full system of these equations is linearized about a spherically symmetric scalar field configuration, which may be thought of as some exact solution. In doing so, we do not assume that the scalar field is small, and do not specify the type of the basic spherically symmetric configuration, which can be a black hole, a naked singularity, or a wormhole.

Keywords: scalar field, self-gravitating configuration, slowly rotation, Einstein-Klein-Gordon equations, contracted Bianchi identities, linearization

MSC numbers: 83C10, 83C25

1 Introduction

We study the mathematical structure of the Einstein-Klein-Gordon equations in a stationary axisymmetric spacetime, where stationarity and axisymmetry are intuitively associated with the rotation of a scalar field supporting such a configuration. One may suppose that a real scalar field exists in nature as a fundamental constituent of matter. On the other hand, it can be considered as a phenomenological model representing a kind of anisotropic fluid, for example, of dark matter in galaxies [1, 2, 3, 4, 5]. In the latter case, the most attractive property of this dark matter model is that a real scalar field acts on other matter only via the gravitational interaction (the spacetime curvature). Hence, this model is an interesting alternative to the cold dark matter phenomenology (see e.g. [6, 7, 8, 9, 10]).

At present, the centers of galaxies and strongly gravitating objects located in them are ones of the most important targets for astronomical observations. It is currently believed that supermassive black holes are the most likely candidates for the role of the central objects. However, the available data are so far insufficient to identify these objects and even to definitely distinguish between black holes, naked singularities, boson stars, and wormholes [11, 12, 13, 14, 15, 16, 17, 18]. Astronomical observations can be interpreted only on the basis of a specific mathematical model, in the framework of which one should not think of the central objects in galaxies as being in vacuum, since dark matter surrounding the centers of galaxies cannot be ignored. In our approach, dark matter is modeled by a nonlinear scalar field, so that there are uncountable number of degrees of freedom in the choice of the scalar field self-interaction potential.

Our main goal in this article is to obtain the contracted Bianchi identities for a stationary axisymmetric spacetime and then to use them for isolating a complete subsystem of independent equations from the full Einstein-Klein-Gordon system. It turns out that there are three ways to performed this operation; as a result, one obtains five independent equations instead of the seven equations of the original system. Apart from that, we consider slowly rotating configurations. In this case, a first order approximation with respect to the perturbations of the spacetime metric is sufficient. We derive the linearized Einstein-Klein-Gordon equations about a spherically symmetric scalar field configuration, which may be thought of as some exact solution. Slowly rotating scalar field configurations have been investigated in a wide context in a number of articles (see [19, 20, 21, 22] and references within).

The paper is organized as follows. In Section 2, writing the metric of a stationary axisymmetric spacetime in the usual form, we derive the full system of the Einstein-Klein-Gordon equations. We consider both classical and phantom scalar fields. Section 3 is devoted to the mathematical investigation of the contracted Bianchi identities in stationary axisymmetric spacetimes. In this section, we prove three propositions about the reduction of the full system of the Einstein-Klein-Gordon equations. We derive and analyze the linearization of the field equations in Section 4. And finally, Section 5 contains a brief discussion and some concluding remarks concerning some possible applications and observational aspects of studying slowly rotating scalar field configurations.

Throughout this article, we use the geometrical system of units ($G = c = 1$) and adopt the metric signature $(+\,-\,-\,-)$. In tensor notation, we use the summation convention over repeated indices; Latin indices take the values 0,1,2,3 while Greek indices run from 1 to 3. The signs of the Riemann and Ricci tensor fields are defined such that $R^i_{jkl} = \partial_k \Gamma^i_{jl} - \ldots$ and $R_{jl} = R^i_{jil}$, respectively.

2 Field equations

2.1 Action, bases, and notations

Assuming the minimal coupling between curvature and a real scalar field ϕ, we start with the action

$$\Sigma = \frac{1}{8\pi} \int \left(-\frac{1}{2} S + \varepsilon \langle d\phi, d\phi \rangle - 2V(\phi) \right) \sqrt{|g|}\, d^4x \,, \tag{1}$$

where S is the scalar curvature, $\varepsilon = \pm 1$ is the sign of the scalar field kinetic term, $V(\phi)$ is a self-interaction potential of the scalar field, and the angle brackets denote the scalar product induced by the metric. The metric of a general axisymmetric spacetime in the standard form can be written as

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 d\theta^2 - D^2 (d\varphi - \sigma dt)^2, \tag{2}$$

where the metric functions A, B, C, D, σ depend only on t, r, and θ.

For the metric (22), the associated orthonormal basis of vector fields and its dual basis are

$$\mathrm{e}_0 = \frac{1}{A}(\partial_t + \sigma \partial_\phi), \quad \mathrm{e}_1 = \frac{1}{B}\,\partial_r, \quad \mathrm{e}_2 = \frac{1}{C}\,\partial_\theta, \quad \mathrm{e}_3 = \frac{1}{D}\,\partial_\phi,$$

$$\mathrm{e}^0 = A dt, \quad \mathrm{e}^1 = B dr, \quad \mathrm{e}^2 = C d\theta, \quad \mathrm{e}^3 = D(d\phi - \sigma dt).$$

The corresponding orthonormal basis of 2-forms, in terms of which the curvature will be expressed, reads

$$\alpha^1 = \mathrm{e}^0 \wedge \mathrm{e}^1,\ \alpha^2 = \mathrm{e}^0 \wedge \mathrm{e}^2,\ \alpha^3 = \mathrm{e}^0 \wedge \mathrm{e}^3,\ *\alpha^1 = \mathrm{e}^3 \wedge \mathrm{e}^2,\ *\alpha^2 = \mathrm{e}^1 \wedge \mathrm{e}^3,\ *\alpha^3 = \mathrm{e}^2 \wedge \mathrm{e}^1,$$

where $*$ is the Hodge star operator.

It is also convenient to introduce the following notation: the directional derivatives along the basis vector fields will be denoted by the corresponding subscript indices placed in the opposite order in parentheses (that can be omitted in practical calculations). For example,

$$\mathrm{e}_0 \phi \equiv \phi_{(0)} = \frac{1}{A}\,\partial_t \phi\,, \quad \mathrm{e}_0 \mathrm{e}_1 C \equiv C_{(1)(0)} = \frac{1}{A}\,\partial_t \left(\frac{1}{B}\,\partial_r C \right). \tag{3}$$

General expressions for the curvature are given in [23] in a more general context and in other notation. Nevertheless, for the sake of completeness, we present the calculations of the connection components in Appendix A and the curvature components in Appendix B in orthonormal bases of vector fields, 1-forms, and 2-forms associated with the metric. For stationary configurations, there is no dependence of the metric on time, however, we have performed the calculations in the general case for subsequent references.

2.2 Einstein equations

In this article, we are mainly interested in the stationary case when the components of the Einstein tensor $G_{ij} = \mathfrak{R}_{ij} - (1/2)Sg_{ij}$ in the basis $\{\mathrm{e}^i \otimes \mathrm{e}^j\}$ can be easily obtain from the results of Appendix B. The algebraically independent components are

$$G_{00} = -\frac{B_{(2)(2)}}{B} - \frac{C_{(1)(1)}}{C} - \frac{D_{(1)(1)}}{D} - \frac{D_{(2)(2)}}{D} - \frac{B_{(2)}D_{(2)}}{BD} - \frac{C_{(1)}D_{(1)}}{CD} - \frac{\sigma^2_{(1)}D^2}{4A^2} - \frac{\sigma^2_{(2)}D^2}{4A^2}\,, \tag{4}$$

$$G_{11} = \frac{A_{(2)(2)}}{A} + \frac{D_{(2)(2)}}{D} + \frac{A_{(1)}C_{(1)}}{AC} + \frac{A_{(1)}D_{(1)}}{AD} + \frac{A_{(2)}D_{(2)}}{AD} + \frac{C_{(1)}D_{(1)}}{CD} + \frac{\sigma^2_{(1)}D^2}{4A^2} - \frac{\sigma^2_{(2)}D^2}{4A^2}\,, \tag{5}$$

$$G_{22} = \frac{A_{(1)(1)}}{A} + \frac{D_{(1)(1)}}{D} + \frac{A_{(2)}B_{(2)}}{AB} + \frac{A_{(1)}D_{(1)}}{AD} + \frac{A_{(2)}D_{(2)}}{AD} + \frac{B_{(2)}D_{(2)}}{BD} - \frac{\sigma^2_{(1)}D^2}{4A^2} + \frac{\sigma^2_{(2)}D^2}{4A^2}\,, \tag{6}$$

$$G_{33} = \frac{A_{(1)(1)}}{A} + \frac{A_{(2)(2)}}{A} + \frac{B_{(2)(2)}}{B} + \frac{C_{(1)(1)}}{C} + \frac{A_{(2)}B_{(2)}}{AB} + \frac{A_{(1)}C_{(1)}}{AC} - \frac{3\sigma^2_{(1)}D^2}{4A^2} - \frac{3\sigma^2_{(2)}D^2}{4A^2}\,, \tag{7}$$

$$G_{03} = \left(\frac{\sigma_{(1)}D}{2A}\right)_{(1)} + \frac{\sigma_{(1)}D_{(1)}}{A} + \frac{\sigma_{(2)}B_{(2)}D}{2AB} + \left(\frac{\sigma_{(2)}D}{2A}\right)_{(2)} + \frac{\sigma_{(2)}D_{(2)}}{A} + \frac{\sigma_{(1)}C_{(1)}D}{2AC}\,, \tag{8}$$

$$G_{12} = -\frac{A_{(2)(1)}}{A} - \frac{D_{(2)(1)}}{D} + \frac{A_{(1)}B_{(2)}}{AB} + \frac{B_{(2)}D_{(1)}}{BD} + \frac{\sigma_{(1)}\sigma_{(2)}D^2}{2A^2}. \tag{9}$$

Analogously, in the stationary case, we obtain (from Appendix C) the energy-momentum tensor components in the orthonormal basis $\{\mathrm{e}^i \otimes \mathrm{e}^j\}$:

$$
\begin{aligned}
8\pi T_{00}^{(\phi)} &= \varepsilon(\phi_{(1)}^2 + \phi_{(2)}^2) + 2V, && (10)\\
8\pi T_{11}^{(\phi)} &= \varepsilon(\phi_{(1)}^2 - \phi_{(2)}^2) - 2V, && (11)\\
8\pi T_{22}^{(\phi)} &= \varepsilon(-\phi_{(1)}^2 + \phi_{(2)}^2) - 2V, && (12)\\
8\pi T_{33}^{(\phi)} &= \varepsilon(-\phi_{(1)}^2 - \phi_{(2)}^2) - 2V, && (13)\\
8\pi T_{03}^{(\phi)} &= 0, && (14)\\
8\pi T_{12}^{(\phi)} &= 2\varepsilon\phi_{(1)}\phi_{(2)}. && (15)
\end{aligned}
$$

Thus the Einstein equations in the orthonormal basis,

$$G_{ij} = 8\pi T_{ij}^{(\phi)}\,, \tag{16}$$

are completely determined by the expressions (4)–(15): it is necessary to equate their right parts.

2.3 Klein-Gordon equations

The scalar field under consideration obeys the equation $\Box\phi + \varepsilon V'_\phi = 0$ or, in the coordinate form,

$$\frac{1}{\sqrt{|g|}}\partial_i g^{ij}\sqrt{|g|}\partial_j\phi + \varepsilon V'_\phi = 0, \quad \phi = \phi(t, r, \theta). \tag{17}$$

By substituting the metric (22) in equation (17), we find

$$
\begin{aligned}
\frac{1}{\sqrt{|g|}}\partial_i g^{ij}\sqrt{|g|}\partial_j\phi &= \frac{1}{ABCD}\left[\partial_t\frac{ABCD}{A^2}\partial_t\phi - \partial_r\frac{ABCD}{B^2}\partial_r\phi - \partial_\theta\frac{ABCD}{C^2}\partial_\theta\phi\right]\\
&= \frac{1}{BCD}(BCD\phi_{(0)})_{(0)} - \frac{1}{ACD}(ACD\phi_{(1)})_{(1)} - \frac{1}{ABD}(ABD\phi_{(2)})_{(2)}\\
&= \phi_{(0)(0)} - \phi_{(1)(1)} - \phi_{(2)(2)} + \frac{(BCD)_{(0)}}{BCD}\phi_{(0)} - \frac{(ACD)_{(1)}}{ACD}\phi_{(1)} - \frac{(ABD)_{(2)}}{ABD}\phi_{(2)}.
\end{aligned}
$$

Thus equation (17) for the axisymmetric spacetime can be written as

$$
\begin{aligned}
&\phi_{(0)(0)} - \phi_{(1)(1)} - \phi_{(2)(2)}\\
&\qquad + \frac{(BCD)_{(0)}}{BCD}\phi_{(0)} - \frac{(ACD)_{(1)}}{ACD}\phi_{(1)} - \frac{(ABD)_{(2)}}{ABD}\phi_{(2)} + \varepsilon V'_\phi = 0.
\end{aligned}
$$

In the stationary case, it takes the form

$$\phi_{(1)(1)} + \phi_{(2)(2)} + \frac{(ACD)_{(1)}}{ACD}\phi_{(1)} + \frac{(ABD)_{(2)}}{ABD}\phi_{(2)} - \varepsilon V'_\phi = 0. \tag{18}$$

3 Bianchi identities and independent equations

In the stationary axisymmetric spacetime, we have the six algebraically independent Einstein equations (16) and the Klein-Gordon equation (18) for the six unknown functions, namely, the field ϕ and the five metric functions A, B, C, D, σ. Moreover, we should impose one coordinate condition on the metric functions, so that the number of unknown functions is effectively five. This means that some two equations in the system of seven equations (16) and (18) are (differential) consequences of the others, so that we should reduce the whole system of the seven equations to a complete independent subsystem of five equations. It turns out that this procedure can be performed in three ways. In order to isolate independent equations, it is necessary to take into account the contracted Bianchi identities.

Let $\Upsilon = \Upsilon_{ij} \mathrm{e}^i \otimes \mathrm{e}^j$ be a symmetric tensor field that obeys the 'differential conservation law'

$$\Upsilon_{ij;k} g^{jk} = 0. \tag{19}$$

We have

$$\begin{aligned}
\Upsilon_{10;0} &= -\frac{A_{(1)}}{A}(\Upsilon_{00} + \Upsilon_{11}) - \frac{A_{(2)}}{A}\Upsilon_{12} + \frac{\sigma_{(1)} D}{2A}\Upsilon_{03}, \\
\Upsilon_{11;1} &= (\Upsilon_{11})_{(1)} + 2\frac{B_{(2)}}{B}\Upsilon_{12}, \\
\Upsilon_{12;2} &= (\Upsilon_{12})_{(2)} + \frac{C_{(1)}}{C}(\Upsilon_{11} - \Upsilon_{22}), \\
\Upsilon_{13;3} &= \frac{D_{(1)}}{D}(\Upsilon_{11} - \Upsilon_{33}) + \frac{D_{(2)}}{D}\Upsilon_{12} - \frac{\sigma_{(1)} D}{A}\Upsilon_{03}, \\
\Upsilon_{20;0} &= -\frac{A_{(2)}}{A}(\Upsilon_{00} + \Upsilon_{22}) - \frac{A_{(1)}}{A}\Upsilon_{12} + \frac{\sigma_{(2)} D}{2A}\Upsilon_{03}, \\
\Upsilon_{21;1} &= (\Upsilon_{21})_{(1)} + \frac{B_{(2)}}{B}(\Upsilon_{22} - \Upsilon_{11}), \\
\Upsilon_{22;2} &= (\Upsilon_{22})_{(2)} + 2\frac{C_{(1)}}{C}\Upsilon_{12}, \\
\Upsilon_{23;3} &= \frac{D_{(2)}}{D}(\Upsilon_{22} - \Upsilon_{33}) + \frac{D_{(2)}}{D}\Upsilon_{12} - \frac{\sigma_{(2)} D}{A}\Upsilon_{03}.
\end{aligned}$$

The substitution of these relations into (19) yields the equations

$$\begin{aligned}
&-(\Upsilon_{11})_{(1)} - (\Upsilon_{12})_{(2)} - \frac{A_{(1)}}{A}(\Upsilon_{00} + \Upsilon_{11}) - \frac{C_{(1)}}{C}(\Upsilon_{11} - \Upsilon_{22}) \\
&\qquad - \frac{D_{(1)}}{D}(\Upsilon_{11} - \Upsilon_{33}) - \left(\frac{A_{(2)}}{A} + 2\frac{B_{(2)}}{B} + \frac{D_{(2)}}{D}\right)\Upsilon_{12} + \frac{\sigma_{(1)} D}{A}\Upsilon_{03} = 0,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&-(\Upsilon_{21})_{(1)} - (\Upsilon_{22})_{(2)} - \frac{A_{(2)}}{A}(\Upsilon_{00} + \Upsilon_{22}) - \frac{B_{(2)}}{B}(\Upsilon_{22} - \Upsilon_{11}) \\
&\qquad - \frac{D_{(2)}}{D}(\Upsilon_{22} - \Upsilon_{33}) - \left(\frac{A_{(1)}}{A} + 2\frac{C_{(1)}}{C} + \frac{D_{(1)}}{D}\right)\Upsilon_{12} + \frac{\sigma_{(2)} D}{A}\Upsilon_{03} = 0.
\end{aligned} \tag{21}$$

Let us assume that $\Upsilon = G - 8\pi\mathcal{T}$. Taking into account the identity $G_{ij;k}g^{jk} = 0$, by direct calculation we can find that equations (20) and (21) are equivalent to the equations $2\phi_{(1)}(\Box\phi + \varepsilon V'(\phi)) = 0$ and $2\phi_{(2)}(\Box\phi + \varepsilon V'(\phi)) = 0$, respectively. Next, we suppose that the Klein-Gordon equation (18) holds. Then equations (20) and (21) become identities.

Now we are able to prove the following propositions.

Proposition 1. *In a stationary axisymmetric spacetime defined by the action (1) and the metric (22), the whole system of field equations (16) and (18) is equivalent to the complete independent subsystem consisting of the Klein-Gordon equation (18) and the four Einstein equations* $G_{00} = 8\pi T_{00},\ G_{11} = 8\pi T_{11},\ G_{22} = 8\pi T_{22},\ G_{12} = 8\pi T_{12}$.

In this case, we have $\Upsilon_{00} = 0$, $\Upsilon_{11} = 0$, $\Upsilon_{22} = 0$, and $\Upsilon_{12} = 0$. The identities (20) and (21) now read

$$\begin{aligned} \frac{D_{(1)}}{D}\Upsilon_{33} + \frac{\sigma_{(1)}D}{A}\Upsilon_{03} &= 0, \\ \frac{D_{(2)}}{D}\Upsilon_{33} + \frac{\sigma_{(2)}D}{A}\Upsilon_{03} &= 0. \end{aligned}$$

The determinant of this linear system, $\big(D_{(1)}\sigma_{(2)} - D_{(2)}\sigma_{(1)}\big)/A$, is nonzero, that is, the system has only the trivial solution $\Upsilon_{33} = \Upsilon_{03} = 0$. $\Box$

Proposition 2. *In a stationary axisymmetric spacetime defined by the action (1) and the metric (22), the whole system of field equations (16) and (18) is equivalent to the complete independent subsystem consisting of the Klein-Gordon equation (18) and the four Einstein equations* $G_{00} = 8\pi T_{00},\ G_{22} = 8\pi T_{22},\ G_{33} = 8\pi T_{33},\ G_{03} = 8\pi T_{03}$.

If the four Einstein equations $\Upsilon_{00} = 0$, $\Upsilon_{22} = 0$, $\Upsilon_{33} = 0$, and $\Upsilon_{03} = 0$ hold, then the identities (20) and (21) take the form

$$\begin{aligned} -(\Upsilon_{11})_{(1)} - (\Upsilon_{12})_{(2)} - \frac{(ACD)_{(1)}}{ACD}\Upsilon_{11} - \frac{(AB^2C)_{(2)}}{AB^2C}\Upsilon_{12} &= 0, \\ -(\Upsilon_{21})_{(1)} + \frac{B_{(2)}}{B}\Upsilon_{11} - \frac{(AC^2D)_{(1)}}{AC^2D}\Upsilon_{12} &= 0. \end{aligned}$$

Suppose that $\Upsilon_{11} \neq 0$ and $\Upsilon_{12} \neq 0$. Writing these identities as

$$\begin{aligned} \Upsilon_{11}\Big[\ln\big(\Upsilon_{11}ACD\big)\Big]_{(1)} + \Upsilon_{12}\Big[\ln\big(\Upsilon_{12}AB^2D\big)\Big]_{(2)} &= 0, \\ -\Upsilon_{11}\Big[\ln B\Big]_{(2)} + \Upsilon_{12}\Big[\ln\big(\Upsilon_{12}AC^2D\big)\Big]_{(1)} &= 0, \end{aligned}$$

we see that the determinant of this linear system is nonzero, which gives us the required contradiction. $\Box$

In an analogous manner, we can prove

Proposition 3. *In a stationary axisymmetric spacetime defined by the action (1) and the metric (22), the whole system of field equations (16) and (18) is equivalent to the complete independent subsystem consisting of the Klein-Gordon equation (18) and the four Einstein equations* $G_{00} = 8\pi T_{00},\ G_{11} = 8\pi T_{11},\ G_{22} = 8\pi T_{22},\ G_{33} = 8\pi T_{33}$.

4 Linearization of the field equations

The complexity of nonlinear equations (16) and (18) does not allow one to study them in a general analytical way. In this section, we obtain the linearized field equations about a spherically symmetric background spacetime. Without loss of generality, we can take the metric of a spherically symmetric spacetime in the form

$$ds^2 = A_0^2 dt^2 - B_0^2 dr^2 - C_0^2 d\theta^2 - C_0^2 \sin^2\theta d\varphi^2, \tag{22}$$

and assume that it is an exact solution of the Einstein-Klein-Gordon equations with some distribution Φ_0 of the scalar field; of course, the functions A_0, B_0, C_0, and Φ_0 depend only on the radial coordinate r.

Any static, asymptotically flat, spherically symmetric solution of the Einstein-Klein-Gordon equations can be a black hole, a naked singularity, a regular solution (boson stars), or a wormhole if we are dealing with a phantom scalar field with $\varepsilon = -1$. At present, a number of exact solutions for self-gravitating nonlinear scalar fields are known [24, 25, 26, 27, 29, 30, 31, 32], with both the positive and negative kinetic terms. All of them have been obtained using the so-called 'inverse problem method for a self-gravitating scalar field minimally coupled to gravity' or, in other words, the 'restored potential method' [33, 34, 35, 36, 37]. In order to obtain the required structure of the spacetime geometry in the inner region of a galaxy, we have a significant degree of freedom in choosing the self-interaction potential of a scalar field, or equivalently, in the spatial distribution of the field. Thus, there are wide opportunities for studying linearized equations (16) and (18) near an analytical spherically symmetric solution. It is especially important that there is a general solution of the Einstein-Klein-Gordon equations in the form of quadratures, so that this mathematical technique allows us to examine the problem, in some sense, for all admissible potentials simultaneously. The quadratures are written explicitly in [37] for the coordinate conditions $C_0 = r$ (black holes, naked singularities, and regular solutions) and $B_0 = 1/A_0$ (all the already listed and wormholes).

We will consider the linearization of the full unreduced system (16) and (18), since the complete independent linearized subsystems also can be isolated in accordance with Propositions 1 – 3. In doing so, we have to perturb the metric and the scalar field of the exact spherically symmetric solution. In other words, the unknown functions in equations (16) and (18) should be represented in the form

$$
\begin{aligned}
A &= A_0 + \tau a(r,\theta)\,,\\
B &= B_0 + \tau b(r,\theta)\,,\\
C &= C_0 + \tau c(r,\theta)\,,\\
D &= C_0 \sin\theta + \tau d(r,\theta)\,,\\
\phi &= \Phi_0 + \tau\psi(r,\theta)\,.
\end{aligned}
$$

After substituting these expressions into equations (16) and (18) and restricting them only to linear terms in τ, we obtain for $a(r,\theta)$, $b(r,\theta)$, $c(r,\theta)$, $d(r,\theta)$, $\psi(r,\theta)$ the following equations:

$$
\begin{aligned}
\{00\}:\ & \frac{1}{B_0^3}\left(-6\,\frac{B_0'C_0'}{B_0C_0} + 2\,\frac{{C_0'}^2}{C_0^2} + 4\,\frac{C_0''}{C_0} + 2\,\varepsilon{\Phi_0'}^2\right)b \\
& - \frac{1}{B_0^2C_0}\left(\frac{B_0'C_0'}{B_0C_0} - \frac{{C_0'}^2}{C_0^2} - \frac{C_0''}{C_0} + 2\,\frac{B_0^2}{C_0^2}\right)c \\
& - \frac{1}{\sin\theta B_0^2C_0}\left(\frac{B_0'C_0'}{B_0C_0} - \frac{{C_0'}^2}{C_0^2} - \frac{C_0''}{C_0} + \frac{B_0^2}{C_0^2}\right)d \\
& + 2\,\frac{C_0'}{B_0^3C_0}\,\partial_r b + \frac{1}{B_0^2C_0}\left(\frac{B_0'}{B_0} - \frac{C_0'}{C_0}\right)\partial_r c \\
& + \frac{1}{\sin\theta B_0^2C_0}\left(\frac{B_0'}{B_0} - \frac{C_0'}{C_0}\right)\partial_r d - \frac{\partial_r^2 c}{B_0^2C_0} \\
& - \frac{\partial_\theta^2 b}{B_0C_0^2} - \frac{\partial_\theta^2 d}{\sin\theta\, C_0^3} - \frac{\cot\theta}{B_0C_0^2}\,\partial_\theta b + \frac{\cot\theta}{C_0^3}\,\partial_\theta c \\
& - 2\varepsilon\,\frac{\Phi_0'}{B_0^2}\,\partial_r\psi - 2\left(\frac{dV}{d\Phi}\right)_{\Phi_0}\psi = 0\,,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\{11\}:\ & -2\,\frac{A_0'C_0'}{A_0^2B_0^2C_0}\,a - \frac{1}{B_0^3}\left(4\,\frac{A_0'C_0'}{A_0C_0} + 2\,\frac{{C_0'}^2}{C_0^2} - 2\,\varepsilon{\Phi_0'}^2\right)b \\
& - \frac{1}{B_0^2C_0}\left(\frac{A_0'C_0'}{A_0C_0} + \frac{{C_0'}^2}{C_0^2} - 2\,\frac{B_0^2}{C_0^2}\right)c \\
& - \frac{1}{\sin\theta B_0^2C_0}\left(\frac{A_0'C_0'}{A_0C_0} + \frac{{C_0'}^2}{C_0^2} - \frac{B_0^2}{C_0^2}\right)d \\
& + 2\,\frac{C_0'}{A_0B_0^2C_0}\,\partial_r a + \frac{1}{B_0^2C_0}\left(\frac{A_0'}{A_0} + \frac{C_0'}{C_0}\right)\partial_r c \\
& + \frac{1}{\sin\theta B_0^2C_0}\left(\frac{A_0'}{A_0} + \frac{C_0'}{C_0}\right)\partial_r d + \frac{\partial_\theta^2 a}{A_0C_0^2} + \frac{\partial_\theta^2 d}{\sin\theta\, C_0^3} \\
& + \frac{\cot\theta}{A_0C_0^2}\,\partial_\theta a - \frac{\cot\theta}{C_0^3}\,\partial_\theta c - 2\varepsilon\,\frac{\Phi_0'}{B_0^2}\,\partial_r\psi + 2\left(\frac{dV}{d\Phi}\right)_{\Phi_0}\psi = 0\,,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\{22\}:\ &\left(\frac{A_0' B_0'}{A_0^2 B_0^3} - \frac{A_0''}{A_0^2 B_0^2} - \frac{A_0' C_0'}{A_0^2 B_0^2 C_0}\right) a + \frac{1}{B_0^3}\left(3\,\frac{A_0' B_0'}{A_0 B_0}\right.\\
&\quad \left. - 2\,\frac{A_0' C_0'}{A_0 C_0} - 2\,\frac{A_0''}{A_0} - 2\,\frac{{C_0'}^2}{C_0^2} + 3\,\frac{B_0' C_0'}{B_0 C_0} - 2\,\frac{C_0''}{C_0} - 2\,\varepsilon {\Phi_0'}^2\right) b\\
&\quad - \frac{{C_0'}^2}{B_0^2 C_0^3}\, c - \frac{1}{\sin\theta B_0^2 C_0}\left(\frac{A_0' C_0'}{A_0 C_0} - \frac{B_0' C_0'}{B_0 C_0} + \frac{C_0''}{C_0} + \frac{{C_0'}^2}{C_0^2}\right) d\\
&\quad - \left(\frac{B_0'}{A_0 B_0^3} - \frac{C_0'}{A_0 B_0^2 C_0}\right)\partial_r a - \left(\frac{A_0'}{A_0 B_0^3} + \frac{C_0'}{B_0^3 C_0}\right)\partial_r b\\
&\quad + \frac{C_0'}{B_0^2 C_0^2}\,\partial_r c + \frac{1}{\sin\theta B_0^2 C_0}\left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{C_0'}{C_0}\right)\partial_r d + \frac{\partial_r^2 a}{A_0 B_0^2}\\
&\qquad + \frac{\partial_r^2 d}{\sin\theta B_0^2 C_0} + 2\varepsilon\,\frac{\Phi_0'}{B_0^2}\,\partial_r\psi + \frac{\cot\theta}{A_0 C_0^2}\,\partial_r a + 2\left(\frac{dV}{d\Phi}\right)_{\Phi_0}\psi = 0\,,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\{33\}:\ &\left(\frac{A_0' B_0'}{A_0^2 B_0^3} - \frac{A_0''}{A_0^2 B_0^2}\right) a + \left(3\,\frac{C_0' B_0'}{B_0^4 C_0} - 2(\varepsilon - 1)\,\frac{{\Phi_0'}^2}{B_0^3} + 2\,\frac{{C_0'}^2}{B_0^3 C_0^2}\right.\\
&\left. - 2\,\frac{A_0''}{A_0 B_0^3} - 2\,\frac{C_0''}{B_0^3 C_0} + 3\,\frac{A_0' B_0'}{A_0 B_0^4}\right) b + \left(\frac{{C_0'}^2}{B_0^2 C_0^3} - \frac{C_0''}{B_0^2 C_0^2} + \frac{B_0' C_0'}{B_0^3 C_0^2}\right) c\\
&+ \frac{{C_0'}^2}{\sin\theta B_0^2 C_0^3}\, d - \frac{B_0'}{A_0 B_0^3}\,\partial_r a - \left(\frac{C_0'}{B_0^3 C_0} + \frac{A_0'}{A_0 B_0^3}\right)\partial_r b\\
&- \left(\frac{C_0'}{B_0^2 C_0^2} + \frac{B_0'}{B_0^3 C_0}\right)\partial_r c - \frac{C_0'}{\sin\theta B_0^2 C_0^2}\,\partial_r d + \frac{\partial_r^2 a}{A_0 B_0^2} + \frac{\partial_r^2 c}{B_0^2 C_0}\\
&\qquad + \frac{\partial_\theta^2 a}{A_0 C_0^2} + \frac{\partial_\theta^2 b}{B_0 C_0^2} + 2(\varepsilon - 1)\,\frac{\Phi_0'}{B_0^2}\,\partial_r\psi + 2\left(\frac{dV}{d\Phi}\right)_{\Phi_0}\psi = 0\,,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\{03\}:\ &\frac{\sin\theta C_0}{A_0 B_0^2}\,\partial_r^2\sigma + \left(-\frac{\sin\theta B_0' C_0}{A_0 B_0^3} + 4\,\frac{\sin\theta C_0'}{A_0 B_0^2} - \frac{\sin\theta A_0' C_0}{A_0^2 B_0^2}\right)\partial_r\sigma\\
&\qquad + \frac{\sin\theta}{A_0 C_0}\,\partial_\theta^2\sigma + 3\,\frac{\cos\theta}{A_0 C_0}\,\partial_\theta\sigma = 0\,,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\{12\}:\ &-\frac{\cot\theta C_0'}{B_0 C_0^3}\, c + \frac{\cot\theta}{B_0 C_0^2}\,\partial_r c + \frac{C_0'}{A_0 B_0 C_0^2}\,\partial_\theta a + \left(\frac{\cot\theta}{B_0 C_0^2} + \frac{A_0'}{A_0 B_0^2 C_0}\right)\partial_\theta b\\
&\quad - \frac{\partial_\theta\partial_r a}{A_0 B_0 C_0} + \frac{C_0'}{\sin\theta C_0^3 B_0}\,\partial_\theta d - 2\varepsilon\,\frac{\Phi_0'}{B_0 C_0}\,\partial_\theta\psi - \frac{\partial_\theta\partial_r d}{\sin\theta C_0^2 B_0} = 0\,,
\end{aligned}
\tag{28}
$$

where a prime denotes differentiation of the basic metric functions A_0, B_0, C_0, and Φ_0 with respect to the radial coordinate r.

5 Conclusions

There are some obvious applications of linearized equations (23) – (28). First, it is well known that boson stars, that is, self-gravitating configurations of a scalar field can mimic stellar-mass black holes in the spherically symmetric approximation. At the same time, there are plausible arguments and observational data that black holes born from single stars rotate very slowly, regardless of initial rotation rate at the moment of formations [38]. Therefore, we can expect that the solutions of the linearized equations will correctly describe such slowly rotating configurations. Second, for any type of supermassive configurations mentioned above, the scalar field describes the surrounding dark matter, so that we should take into account the angular momentum of galactic halos and their central parts. In this case, however, one can also expect a slow (differential) rotation of both galactic halos and the strongly gravitating objects at galactic centers [39, 40, 41]. Third, precision measurements and computer simulations of the orbital parameters of S-stars near the center of our Galaxy should take into account the rotational perturbations of the spacetime metric [42, 43, 44, 45].

Note finally that modern observations, especially in dark matter dominated dwarf galaxies, show an approximately constant dark matter density near galactic centers, while the standard n-body simulation shows a negative power law for the central distribution of dark matter. In this connection, we also can hope that this cusp-core problem, as it commonly called [46, 47], has a partial solution in the rotating dark halo model.

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Appendix

A Connection 1-forms

The metric (2) we are dealing with has the form

$$ds^2 = A^2dt^2 - B^2dr^2 - C^2d\theta^2 - D^2(d\varphi - \sigma dt)^2,$$

where the metric functions

$$A = A(t,\, r,\, \theta),\ B = B(t,\, r,\, \theta),\ C = C(t,\, r,\, \theta),\ D = D(t,\, r,\, \theta),\ \sigma = \sigma(t,\, r,\, \theta)$$

depend, in the general case of a nonstationary spacetime, on three coordinates, because the axial symmetry excludes dependence on the coordinate φ; there is also no dependence on the time coordinate in the stationary case.

All calculations below are performed using the Cartan structure equations [23] in the orthonormal bases defined at the beginning of Section 2. We also have used the notation (3) for the directional derivatives. The first structure equation has the form

$$d\mathrm{e}^i + \omega^i_j \wedge \mathrm{e}^j = 0\,,$$

where $\omega^0_\alpha = \omega^\alpha_0$ and $\omega^\alpha_\beta = -\omega^\beta_\alpha$ due to orthonormality of the basis. From this structure equation we find (the explicit expressions for the exterior derivatives $d\mathrm{e}^i$ can be calculated directly in the usual way and are therefore omitted here) the algebraically independent connection 1-forms

$$\omega^0_1 = \frac{A_{(1)}}{A}\mathrm{e}^0 + \frac{B_{(0)}}{B}\mathrm{e}^1 + \frac{\sigma_{(1)}D}{2A}\mathrm{e}^3\,, \quad \omega^0_2 = \frac{A_{(2)}}{A}\mathrm{e}^0 + \frac{C_{(0)}}{C}\mathrm{e}^2 + \frac{\sigma_{(2)}D}{2A}\mathrm{e}^3\,,$$

$$\omega^0_3 = \frac{\sigma_{(1)}D}{2A}\mathrm{e}^1 + \frac{\sigma_{(2)}D}{2A}\mathrm{e}^2 + \frac{D_{(0)}}{D}\mathrm{e}^3\,, \quad \omega^1_2 = \frac{B_{(2)}}{B}\mathrm{e}^1 - \frac{C_{(1)}}{C}\mathrm{e}^2\,,$$

$$\omega^1_3 = \frac{\sigma_{(1)}D}{2A}\mathrm{e}^0 - \frac{D_{(1)}}{D}\mathrm{e}^3\,, \quad \omega^2_3 = \frac{\sigma_{(2)}D}{2A}\mathrm{e}^0 - \frac{D_{(2)}}{D}\mathrm{e}^3\,.$$

B Curvature

From the second structure equation,

$$\frac{1}{2}\,R_{ijkl}\,\mathrm{e}^k \wedge \mathrm{e}^l \;=\; g_{im}\,(d\omega^m_j \;+\; \omega^m_p \wedge \omega^p_j)\,,$$

we obtain (omitting again the direct calculations of $d\omega^i_j$) the curvature components in the orthonormal basis of 2-forms of Section 2.

The independent components can be written as

$$R_{0101} = -\frac{A_{(1)(1)}}{A} + \frac{B_{(0)(0)}}{B} - \frac{A_{(2)}B_{(2)}}{AB} + \frac{3\sigma_{(1)}^2 D^2}{4A^2};$$

$$R_{0202} = -\frac{A_{(2)(2)}}{A} + \frac{C_{(0)(0)}}{C} - \frac{A_{(1)}C_{(1)}}{AC} + \frac{3\sigma_{(2)}^2 D^2}{4A^2};$$

$$R_{0303} = \frac{D_{(1)(1)}}{D} - \frac{A_{(1)}D_{(1)}}{AD} - \frac{A_{(2)}D_{(2)}}{AD} - \frac{\sigma_{(1)}^2 D^2}{4A^2} - \frac{\sigma_{(2)}^2 D^2}{4A^2};$$

$$R_{2121} = \frac{B_{(2)(2)}}{B} + \frac{C_{(1)(1)}}{C} - \frac{B_{(0)}C_{(0)}}{BC}; \quad R_{2323} = \frac{C_{(1)}D_{(1)}}{CD} + \frac{\sigma_{(2)}^2 D^2}{4A^2};$$

$$R_{1313} = \frac{D_{(1)(1)}}{D} - \frac{B_{(0)}D_{(0)}}{BD} + \frac{B_{(2)}D_{(2)}}{BD} + \frac{\sigma_{(1)}^2 D^2}{4A^2};$$

$$R_{0102} = -\frac{A_{(2)(1)}}{A} + \frac{A_{(1)}B_{(2)}}{AB} + \frac{3\sigma_{(1)}\sigma_{(2)} D^2}{4A^2};$$

$$R_{0103} = \left(\frac{\sigma_{(1)}D}{2A}\right)_{(0)} + \frac{\sigma_{(1)}D_{(0)}}{A}; \quad R_{0203} = \left(\frac{\sigma_{(2)}D}{2A}\right)_{(0)} + \frac{\sigma_{(2)}D_{(0)}}{A};$$

$$R_{0132} = -\left(\frac{\sigma_{(1)}D}{2A}\right)_{(2)} - \frac{\sigma_{(1)}D_{(2)}}{2A} - \frac{\sigma_{(2)}D_{(1)}}{2A} + \frac{\sigma_{(2)}C_{(1)}D}{2AC};$$

$$R_{0113} = \left(\frac{\sigma_{(1)}D}{2A}\right)_{(1)} + \frac{\sigma_{(1)}D_{(1)}}{A} + \frac{\sigma_{(2)}B_{(2)}D}{2AB};$$

$$R_{0121} = \frac{B_{(2)(0)}}{B} - \frac{A_{(2)}B_{(0)}}{AB}; \quad R_{0221} = -\frac{C_{(1)(0)}}{C} + \frac{A_{(1)}C_{(0)}}{AC};$$

$$R_{0232} = -\left(\frac{\sigma_{(2)}D}{2A}\right)_{(2)} - \frac{\sigma_{(2)}D_{(2)}}{A} - \frac{\sigma_{(1)}C_{(1)}D}{2AC};$$

$$R_{0213} = \left(\frac{\sigma_{(2)}D}{2A}\right)_{(1)} + \frac{\sigma_{(2)}D_{(1)}}{2A} + \frac{\sigma_{(1)}D_{(2)}}{2A} - \frac{\sigma_{(1)}B_{(2)}D}{2AB};$$

$$R_{0313} = \frac{D_{(1)(0)}}{D} - \frac{A_{(1)}D_{(0)}}{AD}; \quad R_{0332} = -\frac{D_{(2)(0)}}{D} + \frac{A_{(2)}D_{(0)}}{AD};$$

$$R_{0321} = \left(\frac{\sigma_{(1)}D}{2A}\right)_{(2)} - \left(\frac{\sigma_{(2)}D}{2A}\right)_{(1)} + \frac{\sigma_{(1)}B_{(2)}D}{2AB} - \frac{\sigma_{(2)}C_{(1)}D}{2AC};$$

$$R_{2113} = \frac{\sigma_{(2)}B_{(0)}D}{2AB}; \quad R_{2132} = \frac{\sigma_{(1)}C_{(0)}D}{2AC};$$

$$R_{3213} = -\frac{D_{(2)(1)}}{D} + \frac{B_{(2)}D_{(1)}}{AD} - \frac{\sigma_{(1)}\sigma_{(2)}D^2}{4A^2}.$$

Calculating the curvature, we have used the following identities:

$$\frac{A_{(1)(2)}}{A}-\frac{A_{(2)}C_{(1)}}{AC}=\frac{A_{(2)(1)}}{A}-\frac{A_{(1)}B_{(2)}}{AB},$$

$$\frac{B_{(0)(2)}}{B}-\frac{B_{(2)}C_{(0)}}{BC}=\frac{B_{(2)(0)}}{B}-\frac{A_{(2)}B_{(0)}}{AB},$$

$$\frac{C_{(0)(1)}}{C}-\frac{B_{(0)}C_{(1)}}{BC}=\frac{C_{(1)(0)}}{C}-\frac{A_{(1)}C_{(0)}}{AC},$$

$$\frac{D_{(0)(2)}}{D}-\frac{C_{(0)}D_{(2)}}{CD}=\frac{D_{(2)(0)}}{D}-\frac{A_{(2)}D_{(0)}}{AD},$$

$$\frac{D_{(0)(1)}}{D}-\frac{B_{(0)}D_{(1)}}{BD}=\frac{D_{(1)(0)}}{D}-\frac{A_{(1)}D_{(0)}}{AD},$$

$$\frac{D_{(1)(2)}}{D}-\frac{B_{(2)}D_{(1)}}{BD}=\frac{D_{(2)(1)}}{D}-\frac{C_{(1)}D_{(2)}}{CD},$$

$$\begin{aligned}\left(\frac{\sigma_{(1)}D}{2A}\right)_{(2)}-\left(\frac{\sigma_{(2)}D}{2A}\right)_{(1)}+\frac{\sigma_{(1)}B_{(2)}D}{2AB}-\frac{\sigma_{(2)}C_{(1)}D}{2AC}&\\ =\frac{\sigma_{(2)}A_{(1)}D}{2A^2}-\frac{\sigma_{(1)}A_{(2)}D}{2A^2}+\frac{\sigma_{(1)}D_{(2)}}{2A}-\frac{\sigma_{(2)}D_{(1)}}{2A}&.\end{aligned}$$

The scalar curvature has the form

$$\begin{aligned}S \;=\; & 2\frac{A_{(1)(1)}}{A}+2\frac{A_{(2)(2)}}{A}-2\frac{B_{(0)(0)}}{B}+2\frac{B_{(2)(2)}}{B}-2\frac{C_{(0)(0)}}{C}+2\frac{C_{(1)(1)}}{C}\\ &-2\frac{D_{(0)(0)}}{D}+2\frac{D_{(1)(1)}}{D}+2\frac{D_{(2)(2)}}{D}+2\frac{A_{(2)}B_{(2)}}{AB}+2\frac{A_{(1)}C_{(1)}}{AC}\\ &+2\frac{A_{(1)}D_{(1)}}{AD}+2\frac{A_{(2)}D_{(2)}}{AD}-2\frac{B_{(0)}C_{(0)}}{BC}-2\frac{B_{(0)}D_{(0)}}{BD}\\ &+2\frac{B_{(2)}D_{(2)}}{BD}-2\frac{C_{(0)}D_{(0)}}{CD}+2\frac{C_{(1)}D_{(1)}}{CD}-\frac{\sigma_{(1)}^2D^2}{2A^2}-\frac{\sigma_{(2)}^2D^2}{2A^2}.\end{aligned}$$

C Energy-momentum tensor

For the Lagrangian of the scalar field in the action (1),

$$\mathcal{L}_\phi=\frac{1}{8\pi}\left(\varepsilon\langle d\phi,d\phi\rangle-2V(\phi)\right),$$

the energy-momentum tensor components in the coordinate basis $\{\partial_t,\partial_r,\partial_\theta,\partial_\varphi\}$ can be calculated by the formula [48]

$$\mathcal{T}^{(\phi)}_{ij}=2\frac{\partial\mathcal{L}_\phi}{\partial g^{ij}}-\mathcal{L}_\phi g_{ij}\,. \tag{29}$$

The direct calculation gives

$$\begin{aligned}
8\pi\mathcal{T}^{(\phi)}_{tt} &= \varepsilon(\phi^2_{(0)}(A^2+\sigma^2D^2)+(\phi^2_{(1)}+\phi^2_{(2)})(A^2-\sigma^2D^2))+2V(A^2-\sigma^2D^2),\\
8\pi\mathcal{T}^{(\phi)}_{rr} &= \varepsilon B^2(\phi^2_{(0)}+\phi^2_{(1)}-\phi^2_{(2)})-2VB^2,\\
8\pi\mathcal{T}^{(\phi)}_{\theta\theta} &= \varepsilon C^2(\phi^2_{(0)}-\phi^2_{(1)}+\phi_{(2)})-2VC^2,\\
8\pi\mathcal{T}^{(\phi)}_{\varphi\varphi} &= \varepsilon D^2(\phi^2_{(0)}-\phi^2_{(1)}-\phi_{(2)})-2VD^2,\\
8\pi\mathcal{T}^{(\phi)}_{tr} &= 8\pi\mathcal{T}^{(\phi)}_{rt}=2\varepsilon AB\phi_{(0)}\phi_{(1)},\\
8\pi\mathcal{T}^{(\phi)}_{t\theta} &= 8\pi\mathcal{T}^{(\phi)}_{\theta t}=2\varepsilon AC\phi_{(0)}\phi_{(2)},\\
8\pi\mathcal{T}^{(\phi)}_{t\varphi} &= 8\pi\mathcal{T}^{(\phi)}_{\varphi t}=-\varepsilon(\phi^2_{(0)}-\phi^2_{(1)}-\phi^2_{(2)})\sigma D^2+2V\sigma D^2,\\
8\pi\mathcal{T}^{(\phi)}_{r\theta} &= 8\pi\mathcal{T}^{(\phi)}_{\theta r}=2\varepsilon BC\phi_{(1)}\phi_{(2)}.
\end{aligned}$$

Then in the orthonormal basis $\{\mathrm{e}^i\otimes\mathrm{e}^j\}$, where e^i are defined in Section 2, the nonzero energy-momentum tensor components are given by

$$\begin{aligned}
8\pi T^{(\phi)}_{00} &= \varepsilon(\phi^2_{(0)}+\phi^2_{(1)}+\phi^2_{(2)})+2V,\\
8\pi T^{(\phi)}_{11} &= \varepsilon(\phi^2_{(0)}+\phi^2_{(1)}-\phi^2_{(2)})-2V,\\
8\pi T^{(\phi)}_{22} &= \varepsilon(\phi^2_{(0)}-\phi^2_{(1)}+\phi_{(2)})-2V,\\
8\pi T^{(\phi)}_{33} &= \varepsilon(\phi^2_{(0)}-\phi^2_{(1)}-\phi_{(2)})-2V,\\
8\pi T^{(\phi)}_{01} &= 8\pi\mathcal{T}^{(\phi)}_{10}=2\varepsilon\phi_{(0)}\phi_{(1)},\\
8\pi T^{(\phi)}_{02} &= 8\pi T^{(\phi)}_{20}=2\varepsilon\phi_{(0)}\phi_{(2)},\\
8\pi T^{(\phi)}_{12} &= 8\pi T^{(\phi)}_{21}=2\varepsilon\phi_{(1)}\phi_{(2)}.
\end{aligned}$$

Note that the components $T^{(\phi)}_{02}$ and $T^{(\phi)}_{03}$ vanish due to the symmetry of the tetrad with respect to tangential directions.