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Covariant series in the normal neighborhood of a submanifold

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Abstract. We consider the covariant series in a some normal neighborhood of a submanifold. Such a neighborhood is a generalization of the normal neighborhood of a point. We discuss how the coefficients of the covariant Taylor series of an arbitrary tensor field can be expressed in terms covariant derivatives of the torsion, Riemann curvature and the field under consideration. We also discuss the algorithm of calculating coefficients of a pseudo-Riemannian metric with respect to the corresponding metric connection without torsion. As an example, we calculate the covariant expansion of the Schwarzschild metric in the normal tubular neighborhood of a circular orbit up to fifth order using the Fermi coordinate system.

Keywords: Riemann manifold, normal neighborhood, covariant series, pseudo-Riemannian metric

MSC numbers: 53A45, 53Bxx, 83-08, 83Cxx

1. Introduction

It is well-known that the construction a special coordinate frame for a specific physical system leads, as a rule, to a significant simplification of the corresponding equations and interpretation of the results. In particular, the Riemann normal coordinates are widely used in modern theories of gravity, especially for solving problems with low spacetime symmetry. In 1923, Veblen and Thomas [1], and also independently Eisenhart [2] in 1926, first pointed out the possibility of expanding metric components into a Taylor series with tensor coefficients in normal coordinates centered at a point. In 1969, Petrov [3] showed that the partial derivatives in the coefficients of a Taylor series can be expressed in terms of the covariant derivatives of a given tensor and of the curvature tensor at the origin of normal coordinates. In this work, an explicit form of the first terms in the expansion of an arbitrary tensor field and, in particular, the spacetime metric was obtained. In the next years, the geometric and physical meaning of normal coordinates in general relativity was considered and refined a lot of times [4, 5, 6, 7, 8]. In these works, in particular, the coefficients for the expansions of the spacetime metric in a Riemann normal coordinate system were derived.

Another direction in studying the expansions in normal coordinates is the use of the Fermi coordinate system in which the results obtained for a normal neighbourhood centered at a point are generalized to the normal tubular neighborhood of a smooth curve. In some simple classes of static spacetimes, the form of spacetime metric in normal coordinates can be constructed exactly. For example, in the (anti) de-Sitter spacetime, Fermi coordinates in the normal tubular neighborhood for the timelike geodesic of a static observer are presented in [9]. In general case, there are a number of approaches to construct Fermi coordinates and find the coefficients of the covariant series. All expansions obtained in [10, 11, 12] assume that the based manifold is equipped with a metric. In the article [12], a closed form for the covariant expansions are found using the early known Metric-integral-theorem by Florides and Synge [13], so that the coefficients of covariant series are expressed in special integral formulae. However, the presence of metric on manifold is not necessary condition for existence of covariant series. It is shown in [4] that one can define covariant series on an arbitrary manifold without metric if an affine connection is given; moreover, it is possible that the connection has nonzero torsion.

The purpose of this article is, first, to describe in detail the structure of covariant Taylor expansions in the case when the covariant derivatives determining them are given at the points of some embedded submanifold. This situation arises not only when the normal Fermi coordinates are introduced into the tubular neighborhood of the particle's world line in general relativity, but in a number of other physical applications [14, 15, 16, 17, 18]. In our approach, we follow the works [4, 19] where relatively simple recurrent expressions are presented for calculating the coefficients of covariant series, an estimate of computational complexity is given, and a computational algorithm is proposed. Second, we give a practically important example of covariant expansions in Fermi coordinates.

From the geometrical point of view, expansions in covariant Taylor series are analytic continuation of tensor fields along geodesics by means of the operator of finite parallel transport $\exp(\nabla_X)$. All the expansion formulae obtained below can also be obtained by applying this operator to the tensor field and vector fields of the basis with subsequent expression of the covariant derivatives of the basis vectors in terms of the covariant derivatives of the curvature and torsion tensors at the initial point of the geodesic. However, it is preferable to make a direct transformation of the Taylor series into the corresponding covariant series, since then the issue of convergence falls away, and the application of the operator $\exp(\nabla_X)$ to analytic tensor fields is justified.

This article is organized as follows. Section 2 contains mathematical preliminaries. In Section 3 we consider covariant Taylor series for arbitrary tensor field and for the pseudo-Riemanian metric. Section 4 deals with the covariant expansion of the Schwarzschild spacetime metric in the tubular neighborhood of a circular orbit. The explicit formulae for some coefficients of the series are presented in Appendix 1.

2. Concept of a normal neighborhood and related definitions

In what follows, the $(n+m)$-dimensional $(n>0,\ m\geqslant 0)$ manifold H, the linear connection ∇ on H, interpreted in terms of covariant differentiation, and the m-dimensional submanifold $M\subset H$ together with the inclusion map $\imath : M\to H$ are assumed to be analytic. We assume, in addition, that the manifolds H and M are orientable and that M is connected and parallelizable in the class of analytic vector fields.

The indices α, β or γ will take values from 1 to n, the indices a, b from $n+1$ to $n+m$, and the indices i, j, k, l, from 1 to $n+m$ irrespective of whether or not they have subindices. In this article we will use the Einstein notation, i. e. summation is understood throughout over repeated skew indices. For the curvature and torsion tensors, covariant derivatives, and connection coefficients we adopt following standard definitions [20]:

$$R(Z,X,Y)=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z\,,\quad T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y],$$

$$R(\mathrm{e}_i,\mathrm{e}_j,\mathrm{e}_k)=R^l_{ijk}\mathrm{e}_l,\quad T(\mathrm{e}_i,\mathrm{e}_j)=T^k_{ij}\mathrm{e}_k,$$

$$\nabla_k=\nabla_{\mathrm{e}_k},\quad \nabla_i X=X^j_{;i}e_j,\quad \nabla_i e_j=\Gamma^k_{ij}\mathrm{e}_k.$$

Let $[\mathrm{e}_a]^{n+m}_{n+1}$ be a basis of a module of analytic vector fields on M or, briefly, a basis on M. We choose completions to bases $(\mathrm{e}_1,\dots,\mathrm{e}_{n+m})_p$, $p\in M$, in the tangent spaces $\mathbb{T}_p(H)$ (identifying each vector $\mathrm{e}\in\mathbb{T}_p(M)$ with $\mathbb{T}_p(H)$) in such a way that the components of the completing vectors e_α in local coordinates on H

are analytic functions of the local coordinates on M. The possibility of such a choice is ensured by the paracompactness of the manifold H, which implies the existence of a Riemannian metric on H and hence a normal bundle over M that is trivial by virtue of the assumptions made concerning H and M [20]. At each point $p \in M$, the completing vectors $(\mathrm{e}_1, \ldots, \mathrm{e}_{n+m})_p$ define a subspace $\mathbb{N}_p(M)$, where $\mathbb{T}_p(H) = \mathbb{N}_p(M) \oplus \mathbb{T}_p(M)$ is the direct sum, and the natural structure of the vector bundle on the set

$$NM = \bigcup_{p \in M} \mathbb{N}_p(M)$$

is analytic.

Through every point $p \in M$ we describe all possible geodesics in the direction $\mathbb{N}_p(M)$. Suppose there exists a connected neighborhood ω of the zero section in NM and an analytic diffeomorphism $\psi : \Omega \to V \subset H$ such that $\psi(X) = c(1)$, where $X \in \mathbb{N}_p(M)$, and c is the geodesic with initial conditions (p, X). Then on V there are defined n coordinate functions: If $X = X^\alpha \mathrm{e}_\alpha$, then with the point $c(1)$ there is associated a set of components X^α. For the vector fields $\partial/\partial X^\alpha$, we adopt the notation e_α, taking into account the identity of their values at each point $p \in M$ to the corresponding vectors of the original basis in $\mathbb{N}_p(M)$.

For $m = \dim M = 0$, all the previous constructions reduce to the choice of a basis in $\mathbb{T}_p(H)$ and the introduction of normal coordinates in the neighborhood of the point p. For $m > 0$, we extend the basis vector fields e_a defined on M to V by Lie transport along the vector fields e_α. Then everywhere on V

$$[\mathrm{e}_i,\, \mathrm{e}_\alpha] = 0,\ i = 1, \ldots, n+m,\ \alpha = 1, \ldots, n, \tag{1}$$

but, generally speaking, $[\mathrm{e}_a,\ \mathrm{e}_b]$ are nonzero vector fields, since the original basis on M need not be a coordinate one. Thus, on V there is defined a basis $[\mathrm{e}_i]_1^{n+m}$ of analytic vector fields. For the 1-forms of the dual basis we use the notation e^i.

We will define the connected open set $V \supset M$ together with the constructed basis on V a *normal neighborhood of the submanifold* M. This definition directly generalizes the concept of a normal neighborhood of a point on a manifold with linear connection. To test for the existence of a normal neighborhood of a submanifold in specific cases, it is convenient to consider coordinate charts adapted to M of the form (U, x) on H in which U and $U \cap M$ and the first n coordinates are equal to zero on $U \cap M \neq \emptyset$. The local existence of ψ is obvious, and therefore if M is contained in the union of the supports of a finite set of such charts, which is certainly true for compact M, there exists a normal neighborhood $V \supset M$.

Specialization of the coordinates or frame of reference usually gives some additional differential relations. In this case we have the following lemma.

Lemma 1. *The components of the tangent vectors X to the geodesic c on V with initial conditions*

$$(p, X_p), \qquad X_p = (X^\alpha e_\alpha)_p \in N_p M$$

are constant on c, and

$$X^\alpha = (X^\alpha)_p, \quad X^a = 0. \tag{2}$$

Conversely, if on a curve c such that $c(O) = p \in M$ the components of the vectors tangent to c satisfy the conditions (2), then c is a geodesic.

We extend X to a vector field on V, making the assumption that the conditions (2) hold everywhere on V. Then at every point of the geodesic c for all natural μ, ν, all $a = n+1, ..., n+m$, and a smooth tensor field Q on V

$$\nabla^\mu_X X = 0, \quad \nabla^\nu_a \nabla^\mu_X X = 0; \tag{3}$$

$$\nabla^2_X \mathrm{e}_a = R(X, X, e_a) + \nabla_X (T(X, \mathrm{e}_a)), \tag{4}$$

i. e. $\left(e_a\right)_c$ are Jacobi fields along c;

$$(\nabla^\mu_X Q)(\#) = (\nabla^\mu Q)(\#; X, \dots, X), \tag{5}$$

where the number of arguments X on the right is equal to $\#$, and the symbol $\#$ denotes the set of arguments of the tensor field Q.

In addition, at the initial point $p = c(0) \in M$,

$$\nabla_{(\alpha_1 \dots} \nabla_{\alpha_\mu} \mathrm{e}_{\alpha)} = 0. \tag{6}$$

The proof of the lemma 1 is discussed in [4].

3. Covariant Taylor series

3.1 Covariant expansions of an arbitrary tensor field

In a normal neighborhood $V \supset M$ for any point $q \in V$ there exists a unique geodesic c that satisfies the conditions of the Lemma 1 and is such that $q = c(1)$. The position of the point q is fully determined by the set $(p, X_1, \dots, X_n)$, where $p = c(0) \in M$, and X_α are the components of the directing vector $X_p \in \mathbb{N}_p(M)$. Let X be a vector field defined as in the Lemma 1. By virtue of the relations (5), the partial derivatives with respect to X_α in the coefficients of the Taylor series of the function and, in particular, the total contraction of the tensor fields can be replaced by the corresponding covariant derivatives.

In general case, for an arbitrary smooth tensor field Q we have following result:

Theorem 1. *If the point $q \in V$ lies in the region of convergence of the Taylor series at the point p of the components of an analytic tensor field of type $(s,\ r)$, then*

$$(Q^{j_1\dots j_s}_{i_1\dots i_r})_q = \sum_{\sigma+|\mu|+|\nu|\geq 0} \frac{1}{\sigma!} X^{\gamma_1}\dots X^{\gamma_\sigma}(Q^{l_1\dots l_s}_{k_1\dots k_r;\gamma_1\dots\gamma_\sigma})_p \underset{(\mu_1)}{u}{}^{k_1}_{i_1}\dots \underset{(\mu_r)}{u}{}^{k_r}_{i_r}\underset{(\nu_1)}{v}{}^{j_1}_{l_1}\dots \underset{(\nu_s)}{u}{}^{j_s}_{l_s}, \quad (7)$$

where

$$\sigma, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s \geqslant 0, \quad |\mu| = \mu_1 + \dots + \mu_r, \quad |\nu| = \nu_1 + \dots + \nu_s$$

and the elements of the square matrices $\underset{(\mu)}{u}$, $\underset{(\mu)}{v}$ of order $n+m$ are homogeneous polynomials of degree μ in the coordinates X^α and are determined for all

$$\alpha = 1, \dots,\ n,\ a = n+1, \dots,\ n+m,\ i,\ k = 1, \dots, n+m$$

by the relations

$$\underset{(0)}{u}{}^k_i = \underset{(0)}{v}{}^k_i = \delta^k_i; \quad (8)$$

$$\underset{(1)}{u}{}^k_\alpha = \frac{1}{2} X^\beta (T^k_{\beta\alpha})_p\,, \quad \underset{(1)}{u}{}^k_a = X^\beta(\Gamma^k_{\beta a} + T^k_{\beta a})_p\,; \quad (9)$$

$$\underset{(\mu)}{u}{}^k_i = \frac{1}{\mu+\epsilon(i)} \sum_{\sigma=1}^{\mu} \frac{1}{(\sigma-1)!} X^{\alpha_1}\dots X^{\alpha_\sigma}(T^k_{\alpha_1 l\,;\,\alpha_2\dots\alpha_\sigma})_p \underset{(\mu-\sigma)}{u}{}^l_i + \quad (10)$$

$$+\frac{1}{(\mu+\epsilon(i))(\mu+\epsilon(i)-1)} \sum_{\sigma=2}^{\mu} \frac{1}{(\sigma-2)!} X^{\alpha_1}\dots X^{\alpha_\sigma}(R^k_{\alpha_1\alpha_2 l\,;\,\alpha_3\dots\alpha_\sigma})_p \underset{(\mu-\sigma)}{u}{}^l_i\,,$$

$$\mu \geqslant 2\,,\ \epsilon(\alpha) = 1\,,\ \epsilon(a) = 0\,;$$

The proof of the Theorem 1 are represented in [4]. The most difficult step is the computation of the coefficients of monomials in matrices $\underset{(\mu)}{u}{}^k_i$ if it has been done, the monomials in (10) can be isolated by well-known methods. The algorithm of computation of the coefficients of monomials was considered in details in [19].

3.2 Covariant expansions of a pseudo-Riemannian metric

In this section, we will assume that the linear connection on the manifold H is compatible with the given pseudo-Riemannian metric and has vanishing torsion. Since the covariant derivative of the pseudo-Riemannian metric in given connection is equal to zero, we will obtain from the Eqs. (7) the following result:

$$(g_{ik})_q = \sum_{\mu+\nu\geq 0} (g_{jl})_p \underset{(\mu)}{u}{}^j_i \underset{(\nu)}{u}{}^l_k. \quad (11)$$

Firstly, we consider the expansions (11) in a normal neighborhood of the point $p \in H$; $\dim H = N$, $m = 0$. For this case, we can readily represent (9) and (10) in the form

$$\underset{(1)}{u}{}^{k}_{i} = 0\,, \quad \underset{(\mu)}{u}{}^{k}_{i} = \sum_{\sigma_1+\ldots+\sigma_\tau=\mu} h(\sigma_1,\ldots,\sigma_\tau) \underset{(\sigma_1)}{P}{}^{k}_{l_1} \underset{(\sigma_2)}{P}{}^{l_1}_{l_2} \cdots \underset{(\sigma_\tau)}{P}{}^{l_{\tau-1}}_{i}\,, \tag{12}$$

where $\mu \geqslant 2$, $\sigma_1,\ldots\sigma_\tau \geqslant 2$, $1 \leqslant \tau \leqslant [\mu/2]$,

$$\underset{(\sigma)}{P}{}^{k}_{l} = X^{\alpha_1} \ldots X^{\alpha_\sigma} (R^k_{\alpha_1\alpha_2 l;\alpha_3\ldots\alpha_\sigma})_p\,, \quad \sigma \geqslant 2,$$

$$h(\sigma_1,\ldots,\sigma_\tau) = \prod_{r=1}^{\tau} \frac{\xi(\sigma_r+\sigma_{r+1}+\ldots+\sigma_\tau)}{(\sigma_r-2)!}\,, \quad \xi(\sigma) = \frac{1}{\sigma(\sigma+1)}.$$

After substituting (12) in (11) and collecting together the terms of the same order, we obtain the expansion

$$\begin{aligned}(g_{ik})_q = (g_{ik})_p + \sum_{\substack{\sigma_1+\ldots+\sigma_\tau\\+\lambda_1+\ldots+\lambda_\rho\geqslant 2}} h(\sigma_1,\ldots,\sigma_\tau) h(\lambda_1,\ldots,\lambda_\rho) \times \\ \times (g_{i_1} k_1)_p \underset{(\sigma_1)}{P}{}^{i_1}_{i_2} \underset{(\sigma_2)}{P}{}^{i_2}_{i_3} \cdots \underset{(\sigma_\tau)}{P}{}^{i_\tau}_{i} \underset{(\lambda_1)}{P}{}^{k_1}_{k_2} \underset{(\lambda_2)}{P}{}^{k_2}_{k_3} \cdots \underset{(\lambda_\rho)}{P}{}^{k_\rho}_{k}\,,\end{aligned} \tag{13}$$

or, in reduced form,

$$\begin{aligned}(g_{ik})_q = (g_{ik})_p + \\ + \sum_{N=2}^{\infty} \sum_{t_1+\ldots+t_n=N} (X^1)^{t_1} \ldots (X^n)^{t_n} \sum_{\substack{\sigma_1+\ldots+\sigma_\tau\\+\lambda_1+\ldots+\lambda_\rho=N}} h(\sigma_1,\ldots,\sigma_\tau) h(\lambda_1,\ldots,\lambda_\rho) \times \\ \times {\sum}' \{ g_{i_1 k_1} R^{i_1}_{\alpha_1\alpha_2 i_2;\alpha_3\ldots\alpha_{\sigma_1}} R^{i_2}_{\beta_1\beta_2 i_3;\beta_3\ldots\beta_{\sigma_2}} \ldots R^{i_\tau}_{\gamma_1\gamma_2 i;\gamma_3\ldots\gamma_{\sigma_\tau}} \times \\ \times R^{k_1}_{\delta_1\delta_2 k_2;\delta_3\ldots\delta_{\lambda_1}} R^{k_2}_{\epsilon_1\epsilon_2 k_3;\epsilon_3\ldots\epsilon_{\lambda_2}} \ldots R^{k_\rho}_{\zeta_1\zeta_2 k;\zeta_3\ldots\zeta_{\lambda_\rho}} \}_p.\end{aligned} \tag{14}$$

In (13) and (14), we have adopted the conventions

$$\sigma_1,\ldots,\sigma_\tau,\lambda_1,\ldots,\lambda_\rho \geqslant 2;\ 1 \leqslant \tau+\rho \leqslant [N/2]\,,$$

and we allow vanishing of one and only one of the numbers τ, ρ; $\tau = 0$ ($\rho = 0$) means the absence of indices $\sigma_1,\ldots,\sigma_\tau$ (respectively, $\lambda_1,\ldots,\lambda_\rho$) in the sum, and all factors containing at least one of these indices is assumed to be equal to unity; in $\sum'$, the summation is over all sets

$$(\alpha_1,\ldots,\alpha_{\sigma_1},\beta_1,\ldots,\beta_{\sigma_2},\ldots,\gamma_1,\ldots,\gamma_{\sigma_\tau}, \\ \delta_1,\ldots,\delta_{\lambda_1},\epsilon_1,\ldots,\epsilon_{\lambda_2},\ldots,\zeta_1,\ldots,\zeta_{\lambda_\rho})$$

of Greek indices (the total number of indices is N), among which t_1 indices are equal to 1, t_2 indices are equal to 2,..., t_n indices are equal to n.

Secondly, we deals with important special case of covariant expansions of the components of a pseudo-Riemannian metric in a normal neighborhood of a hypersurface, which corresponds to $\dim H = 1+m$, $\dim M = m$, $n = 1$, the relations (11) can be reduced to the form

$$(g_{11})_q = (g_{11})_p,\ (g_{1a})_q = (g_{1a})_p + X^1 (g_{1l})_p \underset{(1)}{S}{}^l_a, \tag{15}$$

$$(g_{ab})_q = (g_{ab})_p + \sum_{N=1}^{\infty} (X^1)^N \sum_{\substack{\sigma_1+\ldots+\sigma_\tau \\ +\lambda_1+\ldots+\lambda_\rho = N}} g(\sigma_1,\ldots,\sigma_\tau) g(\lambda_1,\ldots,\lambda_\rho) \times \tag{16}$$
$$\times (g_{ik})_p \underset{(\sigma_1)}{S}{}^i_{a_1} \underset{(\sigma_2)}{S}{}^{a_1}_{a_2} \cdots \underset{(\sigma_\tau)}{S}{}^{a_{\tau-1}}_{a_\tau} \underset{(\sigma_0)}{S}{}^{a_\tau}_{a} \underset{(\lambda_1)}{S}{}^{k}_{b_1} \underset{(\lambda_2)}{S}{}^{b_1}_{b_2} \cdots \underset{(\lambda_\rho)}{S}{}^{b_{\rho-1}}_{b_\rho} \underset{(\lambda_0)}{S}{}^{b_\rho}_{b},$$

where

$$\sigma_0, \lambda_0 = 0\,,\ 1;\ \sigma_1,\ldots,\sigma_\tau, \lambda_1,\ldots,\lambda_\rho \geq 2\,;\ 1 \leq \tau+\rho \leq [N/2],$$

and we allow the vanishing of both the numbers τ, ρ; $\tau = 0$ ($\rho = 0$) means the absence of the indices $\sigma_1,\ldots,\sigma_\tau$ (respectively, $\lambda_1,\ldots,\lambda_\rho$) in the sum, and all factors containing at least one of these indices are assumed equal to unity;

$$\underset{(0)}{S}{}^i_a = \delta^i_a\,, \qquad \underset{(1)}{S}{}^i_a = (\Gamma^i_{1a})_p\,;$$

$$\underset{(\sigma)}{S}{}^i_a = (R^i_{11a\,;\,1,\ldots,1})_p \qquad (\sigma \text{ indices } 1)\,, \qquad \sigma \geq 2;$$

$$g(\sigma_1,\ldots,\sigma_\tau) = \prod_{r=1}^{\tau} \frac{\eta(\sigma_r+\sigma_{r+1}+\ldots+\sigma_\tau)}{(\sigma_r-2)!}\,, \qquad \eta(\sigma) = \frac{1}{\sigma(\sigma+1)}.$$

It is readily seen that the expansions (15) and (16) contain not only the connection coefficients Γ^i_{1a} but also m independent components of the Riemann tensor and their covariant derivatives along the congruence of geodesics transversal to the hypersurface, taken at the points of this hypersurface.

4. Covariant expansions of the Schwarzschild metric in the normal tubular neighborhood of a circular orbit

In this section we consider in details how to compute the components of covariant series for the metric of the Schwarzschild spacetime in the normal neighborhood of a circle orbit.

Let us consider the metric of a spherically symmetric spacetime

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 d\theta^2 - C^2 \sin^2\theta d\varphi^2, \tag{17}$$

where functions A, B, C depend on only coordinates t and r. It is convenient to use an orthonormal basis of vector fields associated with the metric (17) and corresponding dual basis of 1-forms to study geodesics. In this vector basis metric components become $(g_{ij}) = \mathrm{diag}\{1, -1, -1, -1\}$. These bases are [1]

$$\varepsilon_0 = \frac{1}{A}\,\partial_t, \quad \varepsilon_1 = \frac{1}{B}\,\partial_r, \quad \varepsilon_2 = \frac{1}{C}\,\partial_\theta, \quad \varepsilon_3 = \frac{1}{C\sin\theta}\,\partial_\varphi, \tag{18}$$

and

$$\varepsilon^0 = A\,dt\,, \quad \varepsilon^1 = B\,dr\,, \quad \varepsilon^2 = C\,d\theta\,, \quad \varepsilon^3 = C\sin\theta\,d\varphi. \tag{19}$$

Furthermore, from here some convenient notation will be used: the directional derivatives along the basis vector fields (18) will be denoted by the corresponding subscript indices placed in parentheses (that can be omitted in practical calculations). For example,

$$\varepsilon_0\phi \equiv \phi_{(0)} = \frac{1}{A}\,\partial_t\phi.$$

Using the Cartan method we can find the forms of connection in the bases (18) and (19). The nonzero forms of connection are listed below:

$$\omega^0_1 = \frac{A_{(1)}}{A}\,\varepsilon^0 + \frac{B_{(0)}}{B}\,\varepsilon^1, \quad \omega^0_2 = \frac{C_{(0)}}{C}\,\varepsilon^2, \quad \omega^0_3 = \frac{C_{(0)}}{C}\,\varepsilon^3, \quad \omega^0_\alpha = \omega^\alpha_0,$$
$$\omega^1_2 = -\frac{C_{(1)}}{C}\,\varepsilon^2\,, \quad \omega^1_3 = -\frac{C_{(1)}}{C}\,\varepsilon^3\,, \quad \omega^2_3 = -\frac{\cot\theta}{C}\,\varepsilon^3\,, \quad \omega^\alpha_\beta = -\omega^\beta_\alpha\,,$$

Note that in the static case (when the metric functions A, B, C depend on only radial coordinate r) all directional derivatives of the metric functions along ε_0 vanish. In particular, it is true for the Schwarzschild spacetime.

The geodesic equation $\nabla_U U = 0$ in the basis (19) gives the system of four equations

$$\frac{dU^i}{ds} + \omega^i_j(U)\,U^j = 0 \tag{20}$$

for the components of the 4-velocity

$$U = U^0\varepsilon_0 + U^1\varepsilon_1 + U^2\varepsilon_2 + U^3\varepsilon_3,$$

where

$$U^0 = A\frac{dt}{ds}\,, \quad U^1 = B\frac{dr}{ds}\,, \quad U^2 = C\frac{d\theta}{ds}\,, \quad U^3 = C\sin\theta\frac{d\varphi}{ds}\,.$$

[1] In this section we will use traditional numeration adopted in General Relativity, i. e. indices will run from 0 to 3.

Without loss of generality we can assume that the geodesics under consideration satisfies the following initial conditions:

$$U^2 = 0\,,\ \theta = \pi/2\,.$$

Then geodesic equation (20) for the U^2-component

$$\frac{dU^2}{ds} + \frac{C_{(0)}}{C}\,U^2U^0 + \frac{C_{(1)}}{C}\,U^2U^1 - \frac{\cot\theta}{C}\,U^3U^3 \;=\; 0\,,$$

implies that the geodesic is are entirely in the equatorial plane. The other three equations take the form

$$\frac{dU^0}{ds} \;+\; \frac{A_{(1)}}{A}\,U^0U^1 \;+\; \frac{B_{(0)}}{B}\,U^1U^1 \;+\; \frac{C_{(0)}}{C}\,U^3U^3 \;=\; 0\,, \tag{21}$$

$$\frac{dU^1}{ds} \;+\; \frac{A_{(1)}}{A}\,U^0U^0 \;+\; \frac{B_{(0)}}{B}\,U^0U^1 \;-\; \frac{C_{(1)}}{C}\,U^3U^3 \;=\; 0\,, \tag{22}$$

$$\frac{dU^3}{ds} \;+\; \frac{C_{(0)}}{C}\,U^0U^3 \;+\; \frac{C_{(1)}}{C}\,U^1U^3 \;=\; 0\,. \tag{23}$$

From (21)–(23) we obtain the first integral [17]

$$\left(U^0\right)^2 - \left(U^1\right)^2 - \left(U^3\right)^2 \;=\; \mathrm{k}\,, \quad \mathrm{k} = -1\,,\,0\,,\,1, \tag{24}$$

where the values $k = 1$, $k = 0$ and $k = -1$ correspond to timelike, null and spacelike geodesics respectively. We restrict our attention on timelike orbits.

Now we turn to consideration of circle orbits. Since for any point of a circle orbit the radial coordinate r is equal to constant, then $U^1 = 0$, U^0 and U^3 are constants. The values of U^0 and U^3 can be found from Eqs. (22) and (24). Assuming $k = 1$ and $U^1 = 0$ we obtain

$$\left(U^0\right)^2 \;=\; \frac{\dfrac{C'}{C}}{\dfrac{C'}{C} - \dfrac{A'}{A}} \;=\; \frac{\dfrac{A}{C}}{\dfrac{A}{C} - \dfrac{dA}{dC}}\,, \qquad \left(U^3\right)^2 \;=\; \frac{\dfrac{A'}{A}}{\dfrac{C'}{C} - \dfrac{A'}{A}} \;=\; \frac{\dfrac{dA}{dC}}{\dfrac{A}{C} - \dfrac{dA}{dC}}\,. \tag{25}$$

Now we describe in details how to receive all terms of the covariant series for the Schwarzschild spacetime metric up to N-th order. Firstly, to solve this task we must specify the Fermi basis on the orbit. From geometrical point of view the circle orbit is one-dimensional submanifold embedded in four-dimensional spacetime, and the values of dimensions m and n introduced in Sec. 2 are equal to 1 and 3, respectively. So, a unit vector that tangent to the orbit must be included in the Fermi basis. This vector can be written in the form

$$\mathrm{e}_0 = U^0\varepsilon_0 + U^3\varepsilon_3, \tag{26}$$

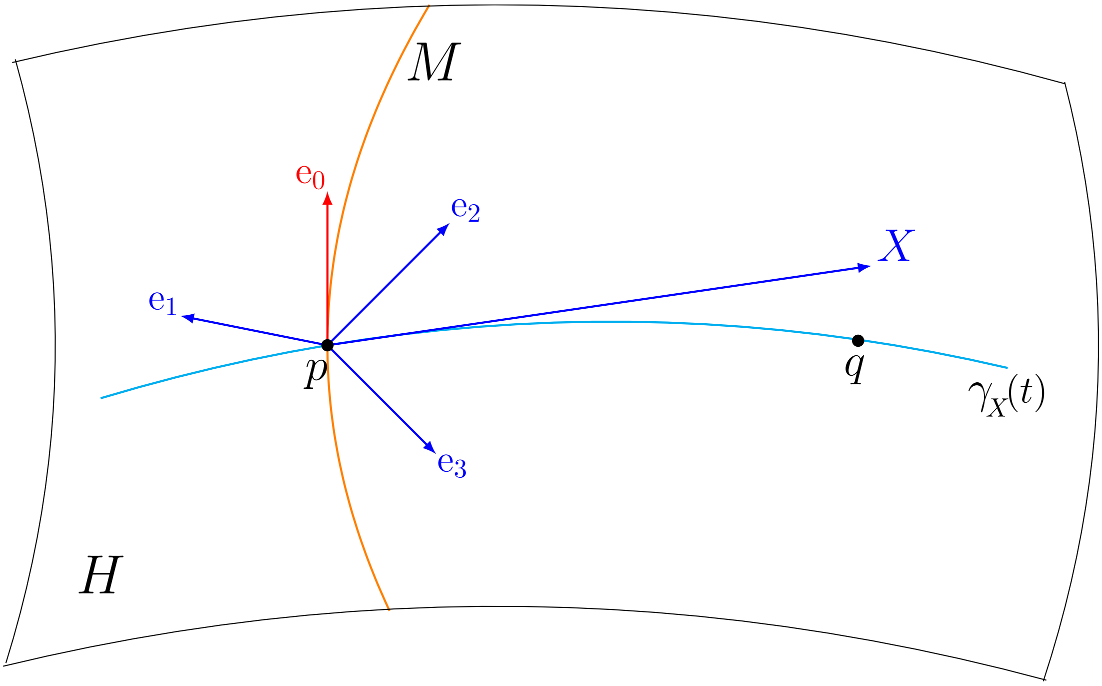

Figure 1: A scheme for covariant expansions along a curve M which is considered as a one-dimensional submanifold in H, $\dim M = 1$, where H is spacetime (four-dimensional manifold). Vector e_0 is tangent to M, and $\mathbb{N}_p M = \langle \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3 \rangle$, $\dim N_p M = 3$. Here γ_X is a geodesic passing through a point $p \in M$ with the tangent vector $X \in \mathbb{N}_p M$, $\gamma_X(1) = q$, so that $X = X^1 \mathrm{e}_1 + X^2 \mathrm{e}_2 + X^3 \mathrm{e}_3$ and (X^1, X^2, X^3) are the nornal coordinates (Fermi coordinates) of the point $q \in H$. In Sec. 4, the basis $\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\} \subset \mathbb{T}_p H$ is chosen to be orthonormal.

where U^0 and U^3 can be found from (25). The other three vectors we define as

$$\mathrm{e}_1 = \varepsilon_1, \qquad \mathrm{e}_2 = \varepsilon_2, \qquad \mathrm{e}_3 = U^3 \varepsilon_0 + U^0 \varepsilon_3. \tag{27}$$

It is easy to verify that the basis (26)–(27) is orthonormal, as well as the basis (18).

The next step is to calculate the matrices $\underset{(\mu)}{u}$ up to N-th order. This step is the most difficult in the whole problem, because it requires to implement a large number of calculations and arithmetical operations.

Let q is a point in the tubular neighborhood of the orbit. It is shown in Sec. 2 that q can be determined unambiguously by a set (p, X^1, X^2, X^3), where p is a point of the orbit connected with q by a unique geodesic, and X^1, X^2 and X^3 are coordinates of the vector $X \in \mathbb{N}_p M$ (M is the circle orbit under consideration) that is tangent to the geodesic in the basis (27) (see Fig. 1).

It is necessary to note that in this context the Greek indices, α, β and γ, in the Eqs. (7) – (11) run the values from 1 to 3, and the Latin ones, a and b, take the value 0.

As we can see from the Eq. (8), the matrix $\underset{(0)}{u}$ is identity. Now we deal with the matrix $\underset{(1)}{u}$. The first equation of the formula (9) implies that the last three columns of $\underset{(1)}{u}$ contain only zeros because of vanishing torsion. To receive the elements $\underset{(1)}{u}{}^i_0$, it is required to calculate the covariant derivatives $\nabla_1 \mathrm{e}_0$, $\nabla_2 \mathrm{e}_0$, $\nabla_3 \mathrm{e}_0$. They are

$$\nabla_1 \mathrm{e}_0 = \nabla_2 \mathrm{e}_0 = 0, \qquad \nabla_3 \mathrm{e}_0 = U^0 U^3 \left(\frac{A_{(1)}}{A} - \frac{C_{(1)}}{C} \right) \mathrm{e}_1.$$

So, there exists a unique nonzero component of the connection $(\Gamma^i_{\alpha 0})_p$ in the Eqs. (9)–(10):

$$\left(\Gamma^1_{30}\right)_p = U^0 U^3 \left(\frac{A_{(1)}}{A} - \frac{C_{(1)}}{C} \right), \qquad \underset{(1)}{u}{}^1_0 = \Gamma^1_{30} X^3.$$

In particular, for the Schwarzschild spacetime $A^2 = (r-2m)/r$, $B^2 = r/(r-2m)$, $C = r$ and we have $\underset{(1)}{u}{}^1_0 = -X^3/r^{3/2}$ and the other elements of the matrix under consideration are zeros.

Now we consider how to compute the elements of the matrix $\underset{(2)}{u}$. It is implies from the Eq. (10) that these elements are expressed via the curvature components. The independent nonzero curvature components of the Schwarzschild spacetime taken in an arbitrary point of the orbit have the form[1]:

$$(R^0_{303})_p = (R^1_{212})_p = (R^2_{121})_p = (R^3_{003})_p = \frac{1}{r^3};$$

$$(R^0_{113})_p = (R^0_{232})_p = (R^1_{013})_p = (R^1_{310})_p =$$
$$= (R^2_{032})_p = (R^2_{302})_p = (R^3_{101})_p = (R^3_{220})_p = \frac{3\sqrt{r-2}}{(r-3)r^3};$$

$$(R^0_{202})_p = (R^1_{313})_p = (R^2_{002})_p = (R^3_{131})_p = \frac{1}{(r-3)r^2};$$

$$(R^0_{110})_p = (R^1_{010})_p = (R^2_{332})_p = (R^3_{223})_p = \frac{2r-3}{(r-3)r^3};$$

$$(R^i_{jkl})_p = -(R^i_{jlk})_p.$$

Further, for example, let us compute the element $\underset{(2)}{u}{}^0_0$. Taking into account

[1]Hereinafter we assume that the parameter m (the Schwarzschild mass) is equal to unity.

vanishing torsion of the spacetime we obtain from the Eq. (10):

$$\underset{(2)}{u}{}^0_0 = \frac{1}{(2+\epsilon(0))(2+\epsilon(0)-1)} X^{\alpha_1} X^{\alpha_2} (R^0_{\alpha_1\alpha_2 l})_p \underset{(0)}{u}{}^l_0 = \frac{1}{2} X^{\alpha_1} X^{\alpha_2} (R^0_{\alpha_1\alpha_2 l})_p \delta^l_0$$

$$= \frac{1}{2}\big((X^1)^2 R^0_{110} + X^1 X^2 R^0_{120} + X^2 X^1 R^0_{210} + X^1 X^3 R^0_{130} + X^3 X^1 R^0_{310}$$

$$+ (X^2)^2 R^0_{220} + X^2 X^3 R^0_{230} + X^3 X^2 R^0_{320} + (X^3)^2 R^0_{330}\big)$$

$$= \frac{(2r-3)(X^1)^2}{2(r-3)r^3} - \frac{(X^2)^2}{2(r-3)r^2} - \frac{(X^3)^2}{2r^3}.$$

The other elements can be received by the same way.

The elements of the matrices $\underset{(\mu)}{u}$ $(\mu \geq 3)$ can be also computed by applying the formula (10), but it is required to calculate all covariant derivatives of the curvature up to $(\mu - 2)$-th order previously. Instead of the recurrent formula (10), we also can use explicit expressions from the article [19].

The final step in solving the problem is to apply the formula (11) and then to collect similar terms. Due to this algorithm we have found the covariant exceptions of the Schwarzschild metric up to 5-th order. The results are presented in the Appendix 1.

5. Conclusions

In this article we have described the concept of a normal neighborhood of a submanifold and discuss how to compute covariant Taylor series. It is shown that covariant series can be defined on manifolds with linear connection and, moreover, with nonzero torsion.

As for practical application of the series, this tool can be used in constructing and studying some realistic models of the motion of a particle in General Relativity. In particular, it can be useful in modeling of an interaction of two gravitational configurations.

We can conclude that the problem of computation of a covariant expansion is too difficult in order to implement it by hand even for sufficiently small orders of an expansion and dimension of the manifold under consideration. Thus, the problem arises of constructing efficient algorithms and their computer implementation. Now there exists an algorithm with exponential computational complexity [19]. It is possible that the solution of the problem will be realized on quantum computers in the near future.

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References

[1] Veblen O. and Thomas T. Y. *The geometry of paths.* Trans. Am. Math. Soc. 1923, **25**, pp. 551–608

[2] L.P. Eisenhart *Riemannian Geometry.* Princeton University Press, Princeton, 1926

[3] Petrov A.Z. *New Methods in the general theory of relativity.* Moscow, Nauka, 1966 (in Russian) [English Edition: Petrov A.Z. *Einstein spaces.* New York, Pergamon Press, 1969]

[4] Tsirulev A. N. *Analytic continuation of tensor fields along geodesics by covariant Taylor series.* Theoretical and Mathematical Physics 1995, **102**, 3, pp. 245–250 (https://doi.org/10.1007/BF01017874)

[5] Mueller U., Schubert C. and Ven A. *A closed formula for the Riemann normal coordinate expansion.* Gen. Rel. Grav. 1999, **31**, pp. 1759–1768 (arXiv:gr-qc/9712092)

[6] Kiyoshi Higashijima, Etsuko Itou and Muneto Nitta *Normal Coordinates in Kähler Manifolds and the Background Field Method.* Progress of Theoretical Physics 2002, **108**, pp. 185–202 (https://doi.org/10.1143/PTP.108.185)

[7] B.Z. Iliev *Handbook of Normal Frames and Coordinates.* Berlin, Birkhäuser Verlag, 2006

[8] L. Brewin *Riemann normal coordinate expansions using Cadabra.* Class. Quantum Grav. 2009, **26**, 175017 (https://doi.org/10.1088/0264-9381/26/17/175017)

[9] D. Klein and P. Collas *Exact Fermi coordinates for a class of space-times.* J. Math. Phys. 2010, **51**, 022501 (https://doi.org/10.1063/1.3298684)

[10] Manasse F.K. and Misner C.W. *Fermi normal coordinates and some basic concepts in differential geometry.* J. Math. Phys. 1963, **4**, pp. 735–745

[11] Karl-Peter Marzlin , 1994 *On the physical meaning of Fermi coordinates.* General Relativity and Gravitation 1994, **26**, pp. 619–636 (https://doi.org/10.1007/BF02108003)

[12] Mukhopadhyay P. *All order covariant tubular expansion.* Reviews in Mathematical Physics 2013, **26**, 1, 1350019 (arXiv:gr-qc/1203.1151)

[13] P.S. Florides and J.L. Synge *Coordinate Conditions in a Riemannian Space for Coordinates Based on a Subspace.* Proceedings of the Royal Society of London, May 25, 1971 — Series A, Mathematical and Physical Sciences, **323**, No. 1552, pp. 1–10 (http://www.jstor.org/stable/77913)

[14] N.D. Birrell and P. C. W. Davies *Quantum Fields in Curved Space.* Cambridge, C.U.P., 1982

[15] E. Poisson, A. Pound and I. Vega *The Motion of Point Particles in Curved Spacetime.* Living Rev. Relativ. 2011, **14**, 7, 163 pp. (https://doi.org/10.12942/lrr-2011-7)

[16] M.N.R. Wohlfarth and C. Pfeifer *Local spacetime effects on gyroscope systems.* Phys. Rev. D 2013, **87**, 024031 (https://doi.org/10.1103/PhysRevD.87.024031)

[17] Nikonov V.V., Potashov I.M. and Tsirulev A.N. *Circular orbits around static self-gravitating scalar field configurations.* Mathematical Modelling and Geometry 2016, **4**, 2, pp. 10–32. (https://mmg/422)

[18] Hari K and Dawood Kothawala *Normal coordinates based on curved tangent space.* Phys. Rev. D 2020, **101**, 124066 (https://doi.org/10.1103/PhysRevD.101.124066)

[19] Potashov I.M. and Tsirulev A.N. *Computational Algorithm for Covariant Series Expansions in General Relativity.* Europ. Phys. J. Web of Conferences 2018, **173**, 03021 (https://doi.org/10.1051/epjconf/201817303021)

[20] Kobayashi S. and Nomizu K. *Foundations of Differential Geometry.* Vols. 1 and 2, New York: Interscience, 1963

Appendix

1. Coefficients of the series for Sec. 4

Here we write out the coefficients up to 5-th order in the basis (26)–(27). They obtain using the algorithm [19] implemented in the computer algebra system Maple.

$$g_{00} = 1 + \frac{(2\,r-3)\,(X^1)^2}{(r-3)\,r^3} - \frac{(X^2)^2}{(r-3)\,r^2} - 2\,\frac{(X^3)^2}{r^3} - \frac{(2\,r-3)\,\sqrt{r-2}(X^1)^3}{r^{9/2}\,(r-3)}$$

$$+\frac{(3\,r^2-8\,r+4)\,X^1\,(X^2)^2}{r^{9/2}\,(r-3)\,\sqrt{r-2}}+3\,\frac{(r^2-3\,r+2)\,X^1\,(X^3)^2}{r^{9/2}\,(r-3)\,\sqrt{r-2}}+\frac{1}{12}\,\frac{(24\,r^2-74\,r+45)\,(X^1)^4}{r^6\,(r-3)}$$

$$-\frac{1}{3}\,\frac{(18\,r^2-50\,r+21)\,(X^1)^2(X^2)^2}{r^6\,(r-3)} - \frac{1}{6}\,\frac{(36\,r^3-220\,r^2+441\,r-297)\,(X^1)^2(X^3)^2}{r^6\,(r-3)^2}$$

$$+\frac{1}{12}\,\frac{(9\,r^2-20\,r-12)\,(X^2)^4}{r^6\,(r-3)} + \frac{1}{2}\,\frac{(11\,r+6-14\,r^2+3\,r^3)\,(X^2)^2(X^3)^2}{r^6\,(r-3)^2}$$

$$+\frac{1}{12}\,\frac{(-15+9\,r^2-10\,r)\,(X^3)^4}{r^6\,(r-3)} - \frac{1}{5}\,\frac{(10\,r^2-28\,r+9)\,\sqrt{r-2}(X^1)^5}{r^{15/2}\,(r-3)}$$

$$+\frac{1}{20}\,\frac{(200\,r^3-963\,r^2+1278\,r-304)\,(X^1)^3(X^2)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$+\frac{1}{20}\,\frac{(-1617\,r^3+4920\,r^2-6724\,r+3504+200\,r^4)\,(X^1)^3(X^3)^2}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}$$

$$-\frac{1}{20}\,\frac{(75\,r^3-330\,r^2+292\,r+136)\,X^1\,(X^2)^4}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$-\frac{3}{20}\,\frac{(813\,r^2-688\,r-352\,r^3+140+50\,r^4)\,X^1\,(X^2)^2(X^3)^2}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}$$

$$-\frac{3}{20}\,\frac{(25\,r^3-117\,r^2+140\,r-12)\,\sqrt{r-2}X^1\,(X^3)^4}{r^{15/2}\,(r-3)^2} + O(X^6);$$

$$g_{01} = \frac{X^3}{r^{3/2}} - 2\,\frac{\sqrt{r-2}X^1\,X^3}{(r-3)\,r^3} + \frac{9}{4}\,\frac{(r-2)\,(X^1)^2X^3}{r^{9/2}\,(r-3)} - \frac{1}{12}\,\frac{(13\,r-30)\,(X^2)^2X^3}{r^{9/2}\,(r-3)}$$

$$- \frac{1}{12}\,\frac{(13\,r-18)\,(X^3)^3}{r^{9/2}\,(r-3)} - \frac{1}{5}\,\frac{(12\,r-25)\,\sqrt{r-2}(X^1)^3X^3}{r^6\,(r-3)}$$

$$+\frac{1}{10}\,\frac{\sqrt{r-2}\,(31\,r-63)\,X^1\,(X^2)^2X^3}{r^6\,(r-3)} + \frac{1}{10}\,\frac{(-193\,r^2+388\,r-252+31\,r^3)\,X^1\,(X^3)^3}{r^6\,(r-3)^2\,\sqrt{r-2}}$$

$$= \frac{1}{4}\,\frac{(10\,r-23)\,(r-2)\,(X^1)^4X^3}{r^{15/2}\,(r-3)} - \frac{1}{360}\,\frac{(2241\,r^2-9040\,r+9186)\,(X^1)^2(X^2)^2X^3}{r^{15/2}\,(r-3)}$$

$$- \frac{1}{360}\,\frac{(2241\,r^3-13990\,r^2+28542\,r-19008)\,(X^1)^2(X^3)^3}{r^{15/2}\,(r-3)^2}$$

$$+ \frac{1}{360}\,\frac{(279\,r^2-1094\,r+1056)\,(X^2)^4X^3}{r^{15/2}\,(r-3)}$$

$$+ \frac{1}{360}\,\frac{(558\,r^3-3313\,r^2+6000\,r-3114)\,(X^2)^2(X^3)^3}{r^{15/2}\,(r-3)^2}$$

$$+ \frac{1}{360}\,\frac{(279\,r^3-1382\,r^2+1932\,r-612)\,(X^3)^5}{r^{15/2}\,(r-3)^2} + O(X^6);$$

$$g_{02} = 2\,\frac{\sqrt{r-2}X^2X^3}{(r-3)r^3} - \frac{1}{6}\,\frac{25r-48)X^1X^2X^3}{r^{9/2}(r-3)} - \frac{1}{5}\,\frac{(7\,r-16)\,\sqrt{r-2}(X^2)^3X^3}{r^6(r-3)}$$

$$+ \frac{1}{10}\,\frac{\sqrt{r-2}(71r-155)(X^1)^2X^2\,X^3}{r^6(r-3)} - \frac{1}{10}\,\frac{(14r^2-59r+39)\sqrt{r-2}X^2(X^3)^3}{r^6(r-3)^2}$$

$$- \frac{1}{180}\,\frac{(1917r^2-8785r+9867)(X^1)^3X^2X^3}{r^{15/2}(r-3)}$$

$$+\frac{1}{180}\,\frac{(1098\,r^2-4913\,r+5442)\,X^1\,(X^2)^3X^3}{r^{15/2}\,(r-3)}$$

$$+\frac{1}{180}\,\frac{(1098\,r^3-7118\,r^2+13863\,r-8073)\,X^1\,X^2\,(X^3)^3}{r^{15/2}\,(r-3)^2}+O(X^6);$$

$$g_{03}=2\,\frac{\sqrt{r-2}(X^1)^2}{(r-3)\,r^3}-2\,\frac{\sqrt{r-2}(X^2)^2}{(r-3)\,r^3}-\frac{9}{4}\,\frac{(r-2)\,(X^1)^3}{r^{9/2}\,(r-3)}+\frac{21}{4}\,\frac{(r-2)\,X^1\,(X^2)^2}{r^{9/2}\,(r-3)}$$

$$+\frac{1}{12}\,\frac{(13\,r-18)\,X^1\,(X^3)^2}{r^{9/2}\,(r-3)}+\frac{1}{5}\,\frac{(12\,r-25)\,\sqrt{r-2}(X^1)^4}{r^6\,(r-3)}$$

$$-\frac{1}{5}\,\frac{\sqrt{r-2}\,(51\,r-109)\,(X^1)^2(X^2)^2}{r^6\,(r-3)}$$

$$-\frac{1}{10}\,\frac{(-193\,r^2+388\,r-252+31\,r^3)\,(X^1)^2(X^3)^2}{r^6\,(r-3)^2\,\sqrt{r-2}}+\frac{1}{5}\,\frac{(7\,r-16)\,\sqrt{r-2}(X^2)^4}{r^6\,(r-3)}$$

$$+\frac{1}{10}\,\frac{(14\,r^2-59\,r+39)\,\sqrt{r-2}(X^2)^2(X^3)^2}{r^6\,(r-3)^2}-\frac{1}{4}\,\frac{(10\,r-23)\,(r-2)\,(X^1)^5}{r^{15/2}\,(r-3)}$$

$$+\frac{1}{24}\,\frac{(405\,r^2-1774\,r+1928)\,(X^1)^3(X^2)^2}{r^{15/2}\,(r-3)}$$

$$+\frac{1}{360}\,\frac{(2241\,r^3-13990\,r^2+28542\,r-19008)\,(X^1)^3(X^3)^2}{r^{15/2}\,(r-3)^2}$$

$$-\frac{1}{24}\,\frac{(165\,r^2-728\,r+796)\,X^1\,(X^2)^4}{r^{15/2}\,(r-3)}$$

$$-\frac{1}{360}\,\frac{(2754\,r^3-17549\,r^2+33726\,r-19260)\,X^1\,(X^2)^2(X^3)^2}{r^{15/2}\,(r-3)^2}$$

$$-\frac{1}{360}\,\frac{(279\,r^3-1382\,r^2+1932\,r-612)\,X^1\,(X^3)^4}{r^{15/2}\,(r-3)^2}+O(X^6);$$

$$g_{11} = -1 + \frac{1}{3}\,\frac{(X^2)^2}{r^3} + \frac{1}{3}\,\frac{(X^3)^2}{(r-3)\,r^2} - \frac{1}{2}\,\frac{\sqrt{r-2}X^1\,(X^2)^2}{r^{9/2}} - \frac{1}{2}\,\frac{\sqrt{r-2}X^1\,(X^3)^2}{(r-3)\,r^{7/2}}$$

$$+ \frac{1}{180}\,\frac{(-251+108\,r)\,(X^1)^2(X^2)^2}{r^6} + \frac{1}{180}\,\frac{(108\,r^2-251\,r+48)\,(X^1)^2(X^3)^2}{r^6\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(27\,r-46)\,(X^2)^4}{r^6} - \frac{1}{180}\,\frac{(-81+54\,r^2-92\,r)\,(X^2)^2(X^3)^2}{r^6\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(27\,r^3-46\,r^2-189\,r+324)\,(X^3)^4}{r^6\,(r-3)^2} - \frac{1}{15}\,\frac{\sqrt{r-2}\,(-29+10\,r)\,(X^1)^3(X^2)^2}{r^{15/2}}$$

$$- \frac{1}{15}\,\frac{(10\,r^2-29\,r+12)\,\sqrt{r-2}(X^1)^3(X^3)^2}{r^{15/2}\,(r-3)} + \frac{1}{2}\,\frac{(r-2)^{3/2}\,X^1\,(X^2)^4}{r^{15/2}}$$

$$+ \frac{(r^3-4\,r^2+3\,r+2)\,X^1\,(X^2)^2(X^3)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$+ \frac{1}{10}\,\frac{(-10\,r^2+116\,r-112+5\,r^4-20\,r^3)\,X^1\,(X^3)^4}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}} + O(X^6);$$

$$g_{12} = -\frac{1}{3}\,\frac{X^1\,X^2}{r^3} + \frac{1}{2}\,\frac{\sqrt{r-2}(X^1)^2X^2}{r^{9/2}} + \frac{1}{2}\,\frac{\sqrt{r-2}X^2\,(X^3)^2}{r^{9/2}\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(-251+108\,r)\,(X^1)^3X^2}{r^6} + \frac{1}{180}\,\frac{(27\,r-46)\,X^1\,(X^2)^3}{r^6}$$

$$+ \frac{1}{180}\,\frac{(27\,r^2-316\,r+489)\,X^1\,X^2\,(X^3)^2}{r^6\,(r-3)} + \frac{1}{15}\,\frac{\sqrt{r-2}\,(-29+10\,r)\,(X^1)^4X^2}{r^{15/2}}$$

$$- \frac{1}{2}\,\frac{(r-2)^{3/2}\,(X^1)^2(X^2)^3}{r^{15/2}} - \frac{1}{10}\,\frac{(5\,r^3-50\,r^2+137\,r-114)\,(X^1)^2X^2\,(X^3)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$- \frac{1}{2}\,\frac{(r^2-4\,r+4)\,(X^2)^3(X^3)^2}{r^{15/2}\sqrt{r-2}\,(r-3)} - \frac{1}{10}\,\frac{(-30\,r^2+57\,r-34+5\,r^3)\,X^2\,(X^3)^4}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}} + O(X^6);$$

$$g_{13} = -\frac{1}{3}\,\frac{X^1\,X^3}{(r-3)\,r^2} + \frac{1}{2}\,\frac{\sqrt{r-2}(X^1)^2X^3}{(r-3)\,r^{7/2}} - \frac{1}{2}\,\frac{\sqrt{r-2}(X^2)^2X^3}{r^{9/2}\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(108\,r^2 - 251\,r + 48)\,(X^1)^3X^3}{r^6\,(r-3)} + \frac{1}{180}\,\frac{(27\,r^2 + 224\,r - 570)\,X^1\,(X^2)^2X^3}{r^6\,(r-3)}$$

$$+ \frac{1}{180}\,\frac{(27\,r^3 - 46\,r^2 - 189\,r + 324)\,X^1\,(X^3)^3}{r^6\,(r-3)^2}$$

$$+ \frac{1}{15}\,\frac{(10\,r^2 - 29\,r + 12)\,\sqrt{r-2}(X^1)^4X^3}{r^{15/2}\,(r-3)}$$

$$- \frac{1}{10}\,\frac{(5\,r^3 + 10\,r^2 - 107\,r + 134)\,(X^1)^2(X^2)^2X^3}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$- \frac{1}{10}\,\frac{(-10\,r^2 + 116\,r - 112 + 5\,r^4 - 20\,r^3)\,(X^1)^2(X^3)^3}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}$$

$$+\frac{1}{2}\,\frac{(r^2 - 4\,r + 4)\,(X^2)^4X^3}{r^{15/2}\sqrt{r-2}\,(r-3)} + \frac{1}{10}\,\frac{(-30\,r^2 + 57\,r - 34 + 5\,r^3)\,(X^2)^2(X^3)^3}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}} + O(X^6);$$

$$g_{22} = -1 + \frac{1}{3}\,\frac{(X^1)^2}{r^3} - \frac{1}{3}\,\frac{(X^3)^2\,(2\,r-3)}{(r-3)\,r^3} - \frac{1}{2}\,\frac{\sqrt{r-2}(X^1)^3}{r^{9/2}}$$

$$+ \frac{1}{2}\,\frac{\sqrt{r-2}\,(4\,r-7)\,X^1\,(X^3)^2}{r^{9/2}\,(r-3)} + \frac{1}{180}\,\frac{(-251 + 108\,r)\,(X^1)^4}{r^6}$$

$$- \frac{1}{180}\,\frac{(27\,r - 46)\,(X^1)^2(X^2)^2}{r^6} - \frac{1}{180}\,\frac{(2910 + 729\,r^2 - 2948\,r)\,(X^1)^2(X^3)^2}{r^6\,(r-3)}$$

$$+ \frac{1}{180}\,\frac{(108\,r^2 - 383\,r + 342)\,(X^2)^2(X^3)^2}{r^6\,(r-3)}$$

$$+\frac{1}{180}\,\frac{(108\,r^3-653\,r^2+1176\,r-639)\,(X^3)^4}{r^6\,(r-3)^2}-\frac{1}{15}\,\frac{\sqrt{r-2}\,(-29+10\,r)\,(X^1)^5}{r^{15/2}}$$

$$+\frac{1}{2}\,\frac{(r-2)^{3/2}\,(X^1)^3(X^2)^2}{r^{15/2}}+\frac{1}{30}\,\frac{(-1944+2800\,r-1324\,r^2+205\,r^3)\,(X^1)^3(X^3)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$-\frac{1}{10}\,\frac{(30\,r^3-181\,r^2+364\,r-244)\,X^1\,(X^2)^2(X^3)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$-\frac{1}{10}\,\frac{(428+30\,r^4-251\,r^3-946\,r+748\,r^2)\,X^1\,(X^3)^4}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}+O(X^6);$$

$$g_{23}=\frac{1}{3}\,\frac{X^2\,X^3\,(2\,r-3)}{(r-3)\,r^3}-\frac{(2\,r^2-7\,r+6)\,X^1\,X^2\,X^3}{r^{9/2}\,(r-3)\,\sqrt{r-2}}$$

$$+\frac{1}{180}\,\frac{(702\,r^2-2632\,r+2421)\,(X^1)^2X^2\,X^3}{r^6\,(r-3)}-\frac{1}{180}\,\frac{(108\,r^2-383\,r+342)\,(X^2)^3X^3}{r^6\,(r-3)}$$

$$-\frac{1}{180}\,\frac{(108\,r^3-653\,r^2+1176\,r-639)\,X^2\,(X^3)^3}{r^6\,(r-3)^2}$$

$$-\frac{1}{30}\,\frac{(190\,r^3-1174\,r^2+2389\,r-1602)\,(X^1)^3X^2\,X^3}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$+\frac{1}{5}\,\frac{(15\,r^3-88\,r^2+172\,r-112)\,X^1\,(X^2)^3X^3}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$+\frac{1}{10}\,\frac{(394+30\,r^4-246\,r^3+718\,r^2-889\,r)\,X^1\,X^2\,(X^3)^3}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}+O(X^6);$$

$$g_{33} = -1 + \frac{1}{3}\,\frac{(X^1)^2}{(r-3)\,r^2} - \frac{1}{3}\,\frac{(X^2)^2\,(2\,r-3)}{(r-3)\,r^3} - \frac{1}{2}\,\frac{\sqrt{r-2}(X^1)^3}{(r-3)\,r^{7/2}}$$

$$+ \frac{1}{2}\,\frac{(4\,r^2-13\,r+10)\,X^1\,(X^2)^2}{r^{9/2}\,(r-3)\,\sqrt{r-2}} + \frac{1}{180}\,\frac{(108\,r^2-251\,r+48)\,(X^1)^4}{r^6\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(729\,r^2-2408\,r+1851)\,(X^1)^2(X^2)^2}{r^6\,(r-3)}$$

$$- \frac{1}{180}\,\frac{(27\,r^3-46\,r^2-189\,r+324)\,(X^1)^2(X^3)^2}{r^6\,(r-3)^2} + \frac{1}{180}\,\frac{(108\,r^2-383\,r+342)\,(X^2)^4}{r^6\,(r-3)}$$

$$+ \frac{1}{180}\,\frac{(108\,r^3-653\,r^2+1176\,r-639)\,(X^2)^2(X^3)^2}{r^6\,(r-3)^2}$$

$$- \frac{1}{15}\,\frac{(10\,r^2-29\,r+12)\,\sqrt{r-2}(X^1)^5}{r^{15/2}\,(r-3)}$$

$$+ \frac{1}{30}\,\frac{(205\,r^3-1144\,r^2+2068\,r-1200)\,(X^1)^3(X^2)^2}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$+ \frac{1}{10}\,\frac{(-10\,r^2+116\,r-112+5\,r^4-20\,r^3)\,(X^1)^3(X^3)^2}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}}$$

$$- \frac{3}{10}\,\frac{(10\,r^3-57\,r^2+108\,r-68)\,X^1\,(X^2)^4}{r^{15/2}\sqrt{r-2}\,(r-3)}$$

$$- \frac{1}{10}\,\frac{(-832\,r+360+688\,r^2-241\,r^3+30\,r^4)\,X^1\,(X^2)^2(X^3)^2}{r^{15/2}\,(r-3)^2\,\sqrt{r-2}} + O(X^6).$$