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Static spherically symmetric space-time and nonlinear spinor field

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Abstract. We studied nonlinear spinor fields in the framework of a static spherically symmetric space-time. In doing so we have suggested a rather simple method to solve the corresponding Einstein equations. As an example, a nonlinear spinor field was considered. The corresponding equations were solved analytically and numerically.

Keywords: spinor field, spherically symmetric model

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1. Introduction

Spherically symmetric space-times are invariant under spatial rotation, hence it is a natural choice to describe simple bodies and island-like configurations [1]. In this case metric functions depend on only radial and time coordinates, while in case of a static spherically symmetric space-time there is no time dependence. One of the most celebrated static spherically symmetric solution to the Einstein equations is the Schwarzschild solution.

Static spherically symmetric space-time is widely exploited is general relativity to model several objects such as compact stars, black holes, wormholes etc. In view of it a system of interacting scalar and electromagnetic fields was studied in [2]. Note that in the recent past spinor description of matter and dark energy was used to draw the picture of the evolution of the Universe within the scope of Bianchi type anisotropic cosmological models [3, 4, 5, 6]. It was found that the approach in question gives rise to a variety of solutions depending on the choice of spinor field nonlinearity. Thanks to its sensitivity to gravitational field spinor field brings some unexpected nuances in the behavior of both the spinor and the gravitational fields. Taking this in mind recently we have studied the nonlinear spinor field within the scope of a static spherically symmetric space-time [7, 8, 9]. The stability of static spherically symmetric solutions of Rastall's theory was studied in [10]. In this report we continue our study which in our view gives a simpler method to study and analyze the equations in question.

2. Basic Equation

The action we choose in the form

$$\mathcal{S} = \int \sqrt{-g}\left[\frac{R}{2\kappa} + L_{\rm sp}\right] d\Omega, \tag{1}$$

where $\kappa = 8\pi G$ is Einstein's gravitational constant, R is the scalar curvature, and $L_{\rm sp}$ is the spinor field Lagrangian given by

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\right] - m_{\rm sp}\bar\psi\psi - \lambda F, \tag{2}$$

where λ is the self-coupling constant. The nonlinear term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. Since ψ and $\psi^\star$ (complex conjugate of ψ) have four component each, one can construct $4 \times 4 = 16$

independent bilinear combinations. They are

$$
\begin{aligned}
S &= \bar\psi\psi \qquad \text{(scalar)}, &\text{(3a)}\\
P &= i\bar\psi\gamma^5\psi \qquad \text{(pseudoscalar)}, &\text{(3b)}\\
v^\mu &= (\bar\psi\gamma^\mu\psi) \qquad \text{(vector)}, &\text{(3c)}\\
A^\mu &= (\bar\psi\gamma^5\gamma^\mu\psi) \qquad \text{(pseudovector)}, &\text{(3d)}\\
T^{\mu\nu} &= (\bar\psi\sigma^{\mu\nu}\psi) \qquad \text{(antisymmetric tensor)}, &\text{(3e)}
\end{aligned}
$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$. Invariants, corresponding to the bilinear forms, are

$$
\begin{aligned}
I &= S^2, &\text{(4a)}\\
J &= P^2, &\text{(4b)}\\
I_v &= v_\mu\, v^\mu = (\bar\psi\gamma^\mu\psi)\, g_{\mu\nu}(\bar\psi\gamma^\nu\psi), &\text{(4c)}\\
I_A &= A_\mu\, A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi)\, g_{\mu\nu}(\bar\psi\gamma^5\gamma^\nu\psi), &\text{(4d)}\\
I_T &= T_{\mu\nu}\, T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi)\, g_{\mu\alpha}g_{\nu\beta}(\bar\psi\sigma^{\alpha\beta}\psi). &\text{(4e)}
\end{aligned}
$$

According to the Fierz identity, among the five invariants only I and J are independent as all others can be expressed by them: $I_v = -I_A = I + J$ and $I_T = I - J$. Therefore, we choose the nonlinear term $F = F(K)$ with K taking one of the following expressions: $\{I,\, J,\, I+J,\, I-J\}$.

The spinor field equations corresponding to the Lagrangian (2) are

$$
\begin{aligned}
\imath\gamma^\mu\nabla_\mu\psi - m_{\rm sp}\psi - \mathcal{D}\psi - \imath\mathcal{G}\gamma^5\psi &= 0, &\text{(5a)}\\
\imath\nabla_\mu\bar\psi\gamma^\mu + m_{\rm sp}\bar\psi + \mathcal{D}\bar\psi + \imath\mathcal{G}\bar\psi\gamma^5 &= 0, &\text{(5b)}
\end{aligned}
$$

where we denote $\mathcal{D} = 2\lambda S F_K K_I$ and $\mathcal{G} = 2\lambda P F_K K_J$, with $F_K = dF/dK$, $K_I = dK/dI$ and $K_J = dK/dJ$. In view of (5) it can be shown that

$$
L_{\rm sp} = \lambda\left(2KF_K - F\right). \tag{6}
$$

The energy-momentum tensor of the spinor field is given by

$$
T_\mu^{\ \rho} = \frac{\imath}{4} g^{\rho\nu}\left(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\right) - \delta_\mu^\rho L_{\rm sp}. \tag{7}
$$

In the expressions above ∇_μ denotes covariant derivative of the spinor field [4]

$$
\nabla_\mu\psi = \partial\psi - \Omega_\mu\psi, \quad \nabla_\mu\bar\psi = \partial\bar\psi + \bar\psi\Omega_\mu, \tag{8}
$$

where Ω_μ is the spinor affine connection which can be defined as [4, 11, 12]

$$
\Omega_\mu = \frac{1}{8}\left[\partial_\mu\gamma_\alpha, \gamma^\alpha\right] - \frac{1}{8}\Gamma^\beta_{\mu\alpha}\left[\gamma_\beta, \gamma^\alpha\right], \tag{9}
$$

where $[a, b] = ab - ba$. Here Dirac matrices in the curve space-time γ are connected to the flat space-time Dirac matrices $\bar{\gamma}$ as follows:

$$\gamma_\mu = e_\mu^{(a)} \bar{\gamma}_a, \quad \gamma^\mu = e^\mu_{(a)} \bar{\gamma}^a, \tag{10}$$

where $e^\alpha_{(a)}$ and $e^{(b)}_\beta$ are the tetrad vectors such that:

$$e_\mu^{(a)} e^\mu_{(b)} = \delta^a_b, \quad e_\mu^{(a)} e^\nu_{(a)} = \delta^\mu_\nu, \tag{11}$$

The γ matrices obey the following anti-commutation rules

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}, \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}, \tag{12}$$

and the flat-space-time Dirac matrices obey

$$\bar{\gamma}_a \bar{\gamma}_b + \bar{\gamma}_b \bar{\gamma}_a = 2 \eta_{ab}, \quad \bar{\gamma}^a \bar{\gamma}^b + \bar{\gamma}^b \bar{\gamma}^a = 2 \eta^{ab}. \tag{13}$$

The spinor affine connection can be equivalently defined as

$$\Omega_\mu(x) = \frac{1}{4} g_{\rho\sigma}(x) \Bigl(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta} \Bigr) \gamma^\sigma \gamma^\delta, \tag{14}$$

where $\Gamma^\beta_{\mu\alpha}$ is the Christoffel symbol.

The spherically-symmetric metric we choose in the form

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \tag{15}$$

where the metric functions γ, α, β depend on the spatial variable u only. Since in order to describe the spherically symmetric gravitational field we need only two components of the metric tensor [14], then in (15) it is possible to choose explicitly one of the three metric functions γ, α, β or demand that all these functions satisfy one of the following coordinate conditions [1, 13, 14]:

1. $\alpha = 0$ - polar Gauss coordinate;

2. $\alpha = \gamma$ - isometric coordinate;

3. $\alpha = \beta$ - homogeneous coordinate;

4. $e^\beta = r$ - curvature coordinate. r is the radius of the sphere with $u = \text{const}$. In this case the metric (15) takes the form

$$ds^2 = e^{2\gamma(r)} dt^2 - e^{2\alpha(r)} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \tag{16}$$

5. $\alpha = \gamma + 2\beta$ - harmonic coordinate;

6. $\alpha = -\gamma$ - quasiblobal coordinate.

In should be noted that since we consider the static spherically symmetric configuration, all the field functions should depend on the spatial variable u only.

Let us now define the spinor affine connection for the metric (15). First we write the corresponding Chrystoffel symbols

$$\begin{aligned}
\Gamma_{10}^{0} &= \gamma', \quad \Gamma_{11}^{1} = \alpha', \quad \Gamma_{12}^{2} = \beta', \\
\Gamma_{13}^{3} &= \beta', \quad \Gamma_{33}^{1} = -\beta' e^{2(\beta-\alpha)} \sin^2\theta, \quad \Gamma_{33}^{2} = -\sin\theta\cos\theta, \\
\Gamma_{00}^{1} &= \gamma' e^{2(\gamma-\alpha)}, \quad \Gamma_{22}^{1} = -\beta' e^{2(\beta-\alpha)}, \quad \Gamma_{23}^{3} = \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{17}$$

Taking into account that tetrad are connected with the metric functions in the following way

$$g_{\mu\nu} = e_{\mu}^{(a)} e_{\nu}^{(b)} \eta_{ab}, \tag{18}$$

where $\eta_{ab} = \mathrm{diag}\{1, -1, -1, -1\}$, for the metric (15) we choose the tetrad as follows:

$$e_0^{(0)} = e^{\gamma}, \quad e_1^{(1)} = e^{\alpha}, \quad e_2^{(2)} = e^{\beta}, \quad e_3^{(3)} = e^{\beta}\sin\theta. \tag{19}$$

Then from (10) and (11) for (15) one now finds

$$\gamma_0 = e^{\gamma}\,\bar\gamma_0, \quad \gamma_1 = e^{\alpha}\,\bar\gamma_1, \quad \gamma_2 = e^{\beta}\,\bar\gamma_2, \quad \gamma_3 = e^{\beta}\,\sin\theta\,\bar\gamma_3. \tag{20}$$

Further taking into account that in our case

$$\bar\gamma^0 = \bar\gamma_0, \quad \bar\gamma^1 = -\bar\gamma_1, \quad \bar\gamma^2 = -\bar\gamma_2, \quad \bar\gamma^3 = -\bar\gamma_3,$$

one also finds

$$\gamma^0 = e^{-\gamma}\bar\gamma^0, \quad \gamma^1 = e^{-\alpha}\bar\gamma^1, \quad \gamma^2 = e^{-\beta}\bar\gamma^2, \quad \gamma^3 = \frac{e^{-\beta}}{\sin\theta}\bar\gamma^3. \tag{21}$$

The flat $\bar\gamma$ matrices we choose in the from

$$\begin{aligned}
\bar\gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \\
\bar\gamma^2 &= \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \bar\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}.
\end{aligned}$$

where I is the unit matrix and σ are the Pauli matrices:

$$\begin{aligned}
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-\imath\varphi} \\ \sin\vartheta e^{\imath\varphi} & -\cos\vartheta \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} -\sin\vartheta & \cos\vartheta e^{-\imath\varphi} \\ \cos\vartheta e^{\imath\varphi} & \sin\vartheta \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & \imath e^{-\imath\varphi} \\ -\imath e^{\imath\varphi} & 0 \end{pmatrix}.
\end{aligned} \tag{22}$$

Defining γ^5 as follows:

$$\begin{aligned}\gamma^5 &= -\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123} = 1,\\ \gamma^5 &= -i\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3 \; = \; -i\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3 = \bar\gamma^5,\end{aligned}$$

we obtain

$$\bar\gamma^5 \;=\; \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.$$

Taking into account that the functions α, β and μ depend on only u (x^1) from (14) we find

$$\Omega_0 \;=\; -\frac{1}{2}\gamma'\, e^{(\gamma-\alpha)}\,\bar\gamma^0\bar\gamma^1, \tag{23a}$$

$$\Omega_0 \;=\; 0, \tag{23b}$$

$$\Omega_2 \;=\; \frac{1}{2}\beta'\, e^{(\beta-\alpha)}\,\bar\gamma^2\bar\gamma^1, \tag{23c}$$

$$\Omega_3 \;=\; \frac{1}{2}\beta'\,\sin\theta e^{(\beta-\alpha)}\,\bar\gamma^3\bar\gamma^1 + \frac{1}{2}\cos\theta\bar\gamma^3\bar\gamma^2. \tag{23d}$$

As it was mentioned earlier for a static spherically symmetric space-time all the functions depend on u. Then in view of (5) and (23) we have [7]

$$\begin{aligned}&\imath e^{-\alpha}\bar\gamma^1\psi' + \frac{\imath}{2}\left(\mu' + 2\beta'\right)e^{-\alpha}\bar\gamma^1\psi + \\ &\qquad \frac{\imath}{2}\frac{\cos\theta}{\theta}e^{-\beta}\bar\gamma^2\psi - m_{\rm sp}\psi - \mathcal{D}\psi - \imath\mathcal{G}\gamma^5\psi \;=\; 0,\end{aligned} \tag{24a}$$

$$\begin{aligned}&\imath e^{-\alpha}\bar\psi'\bar\gamma^1 + \frac{\imath}{2}\left(\mu' + 2\beta'\right)e^{-\alpha}\bar\psi\bar\gamma^1 + \\ &\qquad \frac{\imath}{2}\frac{\cos\theta}{\sin\theta}e^{-\beta}\bar\psi\bar\gamma^2 + m_{\rm sp}\bar\psi + \mathcal{D}\bar\psi + \imath\mathcal{G}\bar\psi\gamma^5 \;=\; 0,\end{aligned} \tag{24b}$$

Then from (7) we find the following non-trivial components of the spinor field [7]:

$$
\begin{aligned}
T_0^0 &= T_2^2 = T_3^3 = \lambda\left(F - 2KF_K\right), & \text{(25a)}\\
T_1^1 &= m_{\rm sp}S + \lambda F, & \text{(25b)}\\
T_1^0 &= \frac{1}{4}\frac{\cos\theta}{\sin\theta}e^{(\alpha-\mu-\beta)}A^3, & \text{(25c)}\\
T_0^1 &= -\frac{1}{4}\frac{\cos\theta}{\sin\theta}e^{(-\alpha+\mu-\beta)}A^3, & \text{(25d)}\\
T_2^0 &= -\frac{1}{4}\left(\mu' - \beta'\right)\, e^{(\beta-\alpha-\mu)}A^3, & \text{(25e)}\\
T_0^2 &= \frac{1}{4}\left(\mu' - \beta'\right)\, e^{(\mu-\beta-\alpha)}A^3, & \text{(25f)}\\
T_3^0 &= \frac{1}{4}\left(\mu' - \beta'\right)\, e^{(\beta-\alpha-\mu)}\sin\theta A^2 + \frac{1}{4}e^{-\mu}\cos\theta A^1, & \text{(25g)}\\
T_0^3 &= -\frac{1}{4}\left(\mu' - \beta'\right)\, e^{(\mu-\beta-\alpha)}\frac{1}{\sin\theta}A^2 - \frac{1}{4}e^{\mu-2\beta}\frac{\cos\theta}{\sin^2\theta}A^1. & \text{(25h)}
\end{aligned}
$$

For the invariants of spinor field we find [7]

$$
\begin{aligned}
S' + \left(\mu' + 2\beta'\right)S - 2e^{\alpha}\mathcal{G}A^1 &= 0, & \text{(26a)}\\
P' + \left(\mu' + 2\beta'\right)P + 2e^{\alpha}\left(m_{\rm sp} + \mathcal{D}\right)A^1 &= 0, & \text{(26b)}\\
A^{1\prime} + \left(\mu' + 2\beta'\right)A^1 + \frac{\cos\theta}{\sin\theta}e^{\alpha-\beta}A^2 + 2e^{\alpha}\left(m_{\rm sp} + \mathcal{D}\right)P + 2e^{\alpha}\mathcal{G}S &= 0, & \text{(26c)}\\
A^{2\prime} + \left(\mu' + 2\beta'\right)A^2 - \frac{\cos\theta}{\sin\theta}e^{\alpha-\beta}A^1 &= 0. & \text{(26d)}
\end{aligned}
$$

The foregoing system gives

$$
\left(SS' - PP' + A^1A^{1\prime} + A^2A^{2\prime}\right) + \left(\mu' + 2\beta'\right)\left(S^2 - P^2 + A^{1^2} + A^{2^2}\right) = 0, \tag{27}
$$

with the relation

$$
\left(S^2 - P^2 + A^{1^2} + A^{2^2}\right) = C_0 e^{-2(\mu+2\beta)}. \tag{28}
$$

On the other hand from Fierz identity we know

$$
S^2 + P^2 = -A_\mu A^\mu = -\left(A^{0^2} + A^{1^2} + A^{2^2} + A^{3^2}\right), \tag{29}
$$

Subtraction of (29) from (28) leads to

$$A^{0^2} = -C_0 e^{-2(\mu+2\beta)} - 2P^2 - A^{3^2}, \tag{30}$$

whereas their addition yields

$$A^{0^2} = C_0 e^{-2(\mu+2\beta)} - 2\left(S^2 + A^{1^2} + A^{2^2}\right) - A^{3^2}. \tag{31}$$

To solve the Einstein equations we have to know how the EMT which is a function of K is connected to metric functions. It can be obtained exploiting the system (26). Let us find the expression for K in all four cases.

1. Let us consider the case when $K = I$. In this case we have $\mathcal{G} = 0$. Then from (26a) one finds

$$S' + (\mu' + 2\beta')\, S = 0, \tag{32}$$

with the solution

$$S = C_S e^{-(\mu+2\beta)} \Longrightarrow K = I = S^2 = C_S^2 e^{-2(\mu+2\beta)}. \tag{33}$$

2. If we choose $K = J$, i.e. $\mathcal{D} = 0$, then in case of massless spinor field from(26b) we find

$$P' + (\mu' + 2\beta')\, P = 0, \tag{34}$$

with the solution

$$P = C_P e^{-(\mu+2\beta)} \Longrightarrow K = J = P^2 = C_P^2 e^{-2(\mu+2\beta)}. \tag{35}$$

3. In case of $K = I + J$ for the massless spinor field from (26a) and (26b) one finds

$$S' + (\mu' + 2\beta')\, S - 2e^{\alpha}\mathcal{G}A^1 = 0, \tag{36a}$$

$$P' + (\mu' + 2\beta')\, P + 2e^{\alpha}\mathcal{D}A^1 = 0. \tag{36b}$$

Taking into account that $\mathcal{D}P = \mathcal{G}S$ from (36) one finds

$$K = I + J = S^2 + P^2 = C_+ e^{-2(\mu+2\beta)}. \tag{37}$$

4. Finally, in case of massless spinor field for $K = I - J$ we find

$$S' + (\mu' + 2\beta')\,S + 2e^{\alpha}\mathcal{G}A^1 = 0, \tag{38a}$$

$$P' + (\mu' + 2\beta')\,P + 2e^{\alpha}\mathcal{D}A^1 = 0, \tag{38b}$$

which yields

$$K = I - J = S^2 - P^2 = C_{-}e^{-2(\mu+2\beta)}. \tag{39}$$

Thus we see that in case of a massless spinor field $K = K_0^2 e^{-2(\mu+2\beta)}$ for K taking any of the expressions $\{I,\ J,\ I+J,\ I-J\}$, while this is true for $K = I$ even in case of a massive spinor.

Note that the metric (15) has only nontrivial diagonal components, hence one should set the non-diagonal components of the energy-momentum tensor trivial. This can impose some restrictions either on the metric functions or on the components of the spinor field as well as on both.

The Einstein tensor corresponding the metric (15) possesses only diagonal elements, hence the Einstein equations in this case takes the form

$$\left(2\gamma'\beta' + \beta'^2\right) - e^{2(\alpha-\beta)} = -\kappa T_1^1, \tag{40a}$$

$$\left(\gamma'^2 + \gamma'\beta' - \gamma'\alpha' + \beta'^2 - \beta'\alpha' + \gamma'' + \beta''\right) = -\kappa T_2^2, \tag{40b}$$

$$\left(3\beta'^2 - 2\beta'\alpha' + 2\beta''\right) - e^{2(\alpha-\beta)} = -\kappa T_0^0. \tag{40c}$$

Subtraction of (40a) from (40c) gives [15]

$$\beta'' + \beta'^2 - \alpha'\beta' - \gamma'\beta' = -\frac{\kappa}{2}\left[T_0^0 - T_1^1\right], \tag{41}$$

whereas subtraction of (41) from (40b) yields [15]

$$\gamma'' + \gamma'^2 - \alpha'\gamma' + 2\gamma'\beta' = -\frac{\kappa}{2}\left[2T_2^2 - T_0^0 + T_1^1\right]. \tag{42}$$

For numerical study it is convenient to rewrite the equations (41) and (42) in the Cauchy form:

$$\beta' = \nu, \tag{43a}$$

$$\gamma' = \tau, \tag{43b}$$

$$\nu' + \nu^2 - \alpha'\nu - \nu\tau = -\frac{\kappa}{2}\left[T_0^0 - T_1^1\right] = -\frac{\kappa}{2}\left[m_{\rm sp}S + 2\lambda K F_K\right], \tag{43c}$$

$$\tau' + \tau^2 - \alpha'\tau + 2\nu\tau = -\frac{\kappa}{2}\left[T_0^0 + T_1^1\right] = -\frac{\kappa}{2}\left[m_{\rm sp}S + 2\lambda\left(F - KF_K\right)\right]. \tag{43d}$$

In (43d) we have taken into account that in our case $T_0^0 = T_2^2$.

As we have already found both S and K can be expressed in terms of metric functions. So we have to give the concrete form of K. We also set $K = I = S^2$ as in this case we can consider massive spinor field. Let us assume that $F = S^n$. In this case we have $T_0^0 = \lambda(1-n)S^n = \lambda(1-n)K_0^n e^{-n(\gamma+2\beta)}$ and $T_1^1 = m_{\rm sp}S + \lambda S^n = m_{\rm sp}K_0 e^{-(\gamma+2\beta)} + \lambda K_0^n e^{-n(\gamma+2\beta)}$.

To solve the equations we have to impose some additional conditions which can be given by for example harmonic and quasiglobal coordinate.

Case 1

Let us first consider the harmonic radial coordinate such that $\alpha = \gamma + 2\beta$ [13, 14]. In view of it the Eqns. (43) can be written as

$$\beta' = \nu, \tag{44a}$$

$$\gamma' = \tau, \tag{44b}$$

$$\nu' = \nu^2 + 2\nu\tau - \frac{\kappa}{2}\left[\lambda n K_0^n e^{-n(\gamma+2\beta)} + m_{\rm sp}K_0 e^{-(\gamma+2\beta)}\right], [\tag{44c}$$

$$\tau' = -\frac{\kappa}{2}\left[\lambda(2-n)K_0^n e^{-n(\gamma+2\beta)} + m_{\rm sp}K_0 e^{-(\gamma+2\beta)}\right]. \tag{44d}$$

Let us solve the foregoing system numerically. Our aim is to obtain some qualitative solutions, so for simplicity we set $K_0 = 1$, $\lambda = 1$ and $m_{\rm sp} = 1$. We also set the following initial values for the metric functions: $\tau(0) = 0.05$, $\nu(0) = 0.05$, $\gamma(0) = 0.5$, $\beta(0) = 0.5$. In Fig. 1 we have plotted the metric functions for linear case with $n = 1$, while in Fig. 2 we have considered the nonlinear case with $n = 4$.

Case II

Let us now chose quasiglobal radial coordinate [1] such that $\alpha = -\gamma$. Then from (43) we find

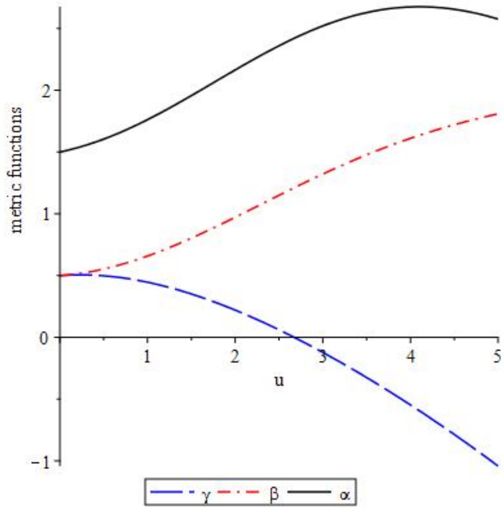

Figure 1: Plot of metric functions for linear spinor field

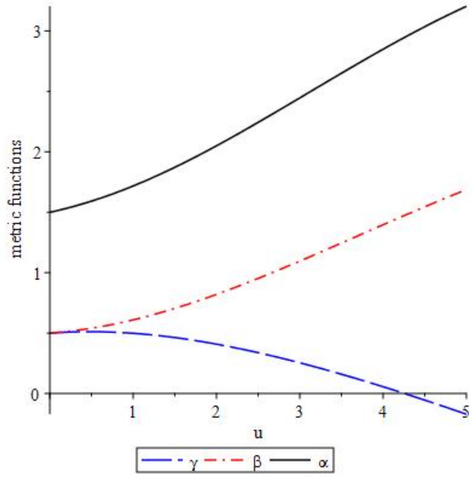

Figure 2: Plot of metric functions for a massive nonlinear spinor field with power on nonlinearity $n = 4$

$$\beta' = \nu, \tag{45a}$$
$$\gamma' = \tau, \tag{45b}$$
$$\nu' = -\nu^2 - \frac{\kappa}{2}\left[\lambda n K_0^n e^{-n(\gamma+2\beta)} + m_{\rm sp} K_0 e^{-(\gamma+2\beta)}\right], \tag{45c}$$
$$\tau' = -2\tau^2 - 2\tau\nu - \frac{\kappa}{2}\left[\lambda(2-n) K_0^n e^{-n(\gamma+2\beta)} + m_{\rm sp} K_0 e^{-(\gamma+2\beta)}\right]. \tag{45d}$$

Like in previous case here too we will solve the system (45) numerically. We consider the same parameters as in previous case. In Fig. 3 we have plotted the metric functions for linear case with $n = 1$, while in Fig. 4 we have considered the nonlinear case with $n = 4$.

As one sees, depending on the choice of coordinate condition we have completely different behavior of the metric functions and the choice of nonlinearity does not qualitatively change the solution, it only gives quantitatively different picture.

3. Conclusion

Within the scope of a static spherically symmetric space-time we have investigated the role of the spinor field nonlinearity. In doing so we have written the Einstein

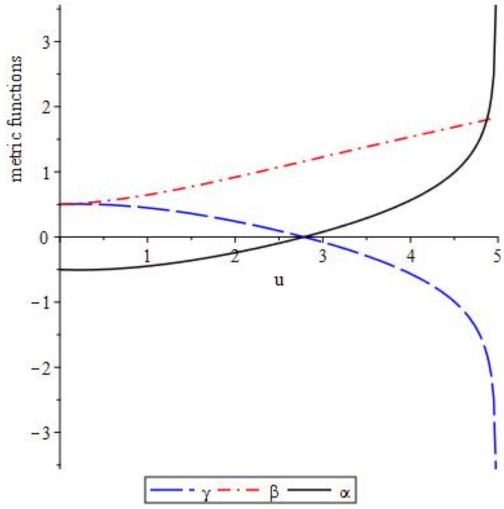

Figure 3: Plot of metric functions for linear spinor field.

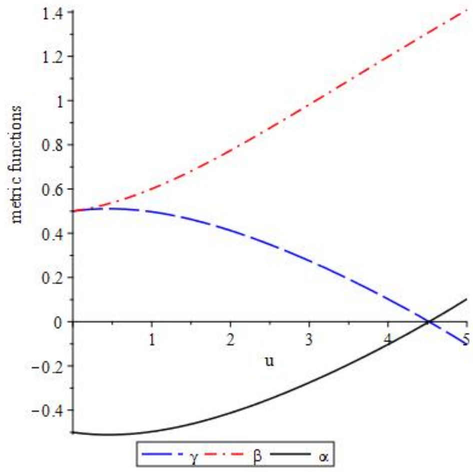

Figure 4: Plot of metric functions for a massive nonlinear spinor field with power on nonlinearity $n = 4$

gravitational field equations in such a way that knowing the energy-momentum tensor of the source field one can have some idea about the type of solution. Though we have exploited the spinor field as the source, it is possible to consider other well known fields and compare the results.

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