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The crossing numbers of several graphs of order eight with paths

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Abstract. The crossing number of a simple graph G is the minimum number of edge crossings in any drawing of G in the plane. There are several classes of graphs for which crossing numbers have been published. One of them is the Cartesian product of two graphs. We give a new results by giving the exact values of crossing numbers of Cartesian product of a few graphs of order eight with paths.

Keywords: graphs, drawings, crossing numbers.

MSC numbers: 05C10; 05C38

1. Introduction

Let G be a simple, undirected and connected graph with set of vertices V and set of edges E. A mapping that assigns a point in the plane for each vertex and for each edge a continuous curve between its two endpoints is called a *drawing* of the graph $G = (V, E)$. A *crossing* of two edges is the intersection of the interiors of the corresponding curves. The *crossing number*, $\mathrm{cr}(G)$, of a graph G is the minimum number of edge crossings in a drawing of G in the plane. The drawing with minimum number of crossings must be a *good drawing*, that means, each two edges have at most one point in common, which is either a common end-vertex or a crossing and no three edges cross at the same point.

The search for formulas for minimal number of crossings was initated by Hungarian mathematician Pal Turan [8]. In 1940 he was sent to labor camp outside Budapest. He worked in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected to all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. He had to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. The work was not difficult but the problem was at the crossings. The trucks jumped the rails, and the bricks fell out from them. He was ask: what is the minimal number of crossings? Turan realized that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult.

In graph theory, we represented the kilns and storage yards by vertices and the tracks by edges and we are asking what is the minimum number of crossings of the complete bipartite graph $K_{m,n}$ which has two sets of vertices, one with m vertices and the other one with n vertices such, that each vertex in one set is joined to every vertex in the other set.

Turan devised a drawing of $K_{m,n}$ with $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor$ crossings, but the conjecture of Zarankiewicz [10] that such a drawing is the best possible, is still opened. The crossing numberof $K_{m,n}$ is proved for every n and for $1 \leq m \leq 6$ [5]. In [9] Woodall publised that the crossing number of $K_{m,n}$ is equal to Zarankiewicz number for $7 \leq m \leq 8$ and $7 \leq n \leq 10$.

Garey and Johnson proved that compute the crossing number for a given graph is NP-complete problem [3]. The problem of determining the crossing number of a given graph has been studied in graph theory and in computer science, VLSI-layout.

Cartesian products of two graphs are one of few graph classes for which the crossing numbers were published. Cartesian product $G_1 \square G_2$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph with vertex set $V = V_1 \times V_2$ and two vertices (v_1, v_2) and (u_1, u_2) are adjacent in $G_1 \square G_2$ if and only if either $v_1 = u_1$ and v_2 is adjacent with u_2 in G_2 or $v_2 = u_2$ and v_1 is adjacent with u_1 in G_1.

Let C_n be the cycle with n edges, P_n be the paths with n edges and S_n be the star with n edges. The crossing numbers for the Cartesian products of some specific graphs G on four, five, six, seven vertices with cycles, paths and stars were studied.

Besides of Cartesian product, there are join and strong products of two graphs for which exact values of crossing numbers were determined.

In this paper we give an exact values of the crossing numbers of Cartesian products of 27 graphs on eigth vertices and eight edges which contain a cycle C_3 (see Figure 1) with paths P_n.

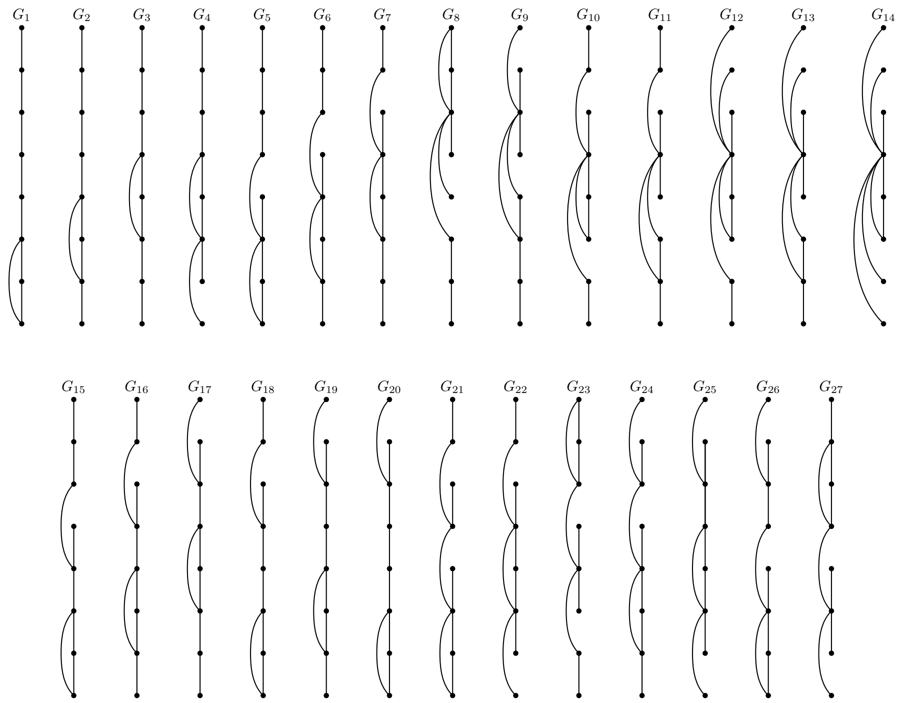

Figure 1: Several 8–vertex graphs on 8 edges with 3–cycle

2. The crossing number of the Cartesian products of some graphs with paths

Every graph $G_i \square P_n$ contains $n+1$ copies of the graph G_i. An uppper bound for the crossing number of the graph $G_i \square P_n$ is a number of crossings in a drawing of $G_i \square P_n$. On the other hand, if the graph $G_i \square P_n$ contains a subgraph for which the crossing number is known yet, we have a lower bound for the crossing number for the graph $G_i \square P_n$.

Theorem 1. *For $n \geq 1$,*

$$\begin{aligned}
&\mathrm{cr}(G_1 \square P_n) = \mathrm{cr}(G_2 \square P_n) = \mathrm{cr}(G_3 \square P_n) = n-1,\\
&\mathrm{cr}(G_4 \square P_n) = \mathrm{cr}(G_5 \square P_n) = \mathrm{cr}(G_6 \square P_n) = \mathrm{cr}(G_7 \square P_n) = 2(n-1),\\
&\mathrm{cr}(G_8 \square P_n) = \mathrm{cr}(G_9 \square P_n) = \mathrm{cr}(G_{10} \square P_n) = \mathrm{cr}(G_{11} \square P_n) = 4(n-1),\\
&\mathrm{cr}(G_{12} \square P_n) = \mathrm{cr}(G_{13} \square P_n) = 6(n-1),\\
&\mathrm{cr}(G_{14} \square P_n) = 9(n-1).
\end{aligned}$$

Proof: In Figures 2(a), 2(b), and 2(c) one can find the drawings of the graphs $G_1\square P_n$, $G_2\square P_n$, and $G_3\square P_n$. The edges of $n-1$ copies of the graphs G_1, G_2, and G_3 are crossed once, so the crossing number of $G_1\square P_n$, $G_2\square P_n$, and $G_3\square P_n$ is at most $n-1$. The drawings of the graphs $G_4\square P_n$, $G_5\square P_n$, $G_6\square P_n$, and $G_7\square P_n$, in which the edges of $n-1$ copies of subgraphs isomorphic to G_4, G_5, G_6, and G_7 are crossed twice are shown in Figures 2(d), 2(e), 2(f), and 2(g). So, $\mathrm{cr}(G_4\square P_n) \leq 2(n-1)$, $\mathrm{cr}(G_5\square P_n) \leq 2(n-1)$, $\mathrm{cr}(G_6\square P_n) \leq 2(n-1)$, and $\mathrm{cr}(G_7\square P_n) \leq 2(n-1)$. The drawings of the graphs $G_8\square P_n$, $G_9\square P_n$, $G_{10}\square P_n$, and $G_{11}\square P_n$ with $4(n-1)$ crossings are shown in Figures 2(h)–2(k). Thus, $\mathrm{cr}(G_8\square P_n) \leq 4(n-1)$, $\mathrm{cr}(G_9\square P_n) \leq 4(n-1)$, $\mathrm{cr}(G_{10}\square P_n) \leq 4(n-1)$, and $\mathrm{cr}(G_{11}\square P_n) \leq 4(n-1)$. There are drawings of the graphs $G_{12}\square P_n$ and $G_{13}\square P_n$ with $6(n-1)$ crossings (see Figure 2(l) and 2(m)). So, $\mathrm{cr}(G_{12}\square P_n) \leq 6(n-1)$ and $\mathrm{cr}(G_{13}\square P_n) \leq 6(n-1)$. There is a drawing of the graph $G_{14}\square P_n$ with $9(n-1)$ crossings (see Figure 2(n)). So, $\mathrm{cr}(G_{14}\square P_n) \leq 9(n-1)$.

Now, we find the lower bounds of crossing numbers $\mathrm{cr}(G_i\square P_n)$ for $i = 1, \ldots, 14$. The graphs G_1, G_2, and G_3 contain the graph S_3 as a subgraph. So, the Cartesian products $G_1\square P_n$, $G_2\square P_n$, and $G_3\square P_n$ contain $S_3\square P_n$ as a subgraph. It was proved in [4] that $\mathrm{cr}(S_3\square P_n) = n-1$. It implies, that the crossing number of $G_1\square P_n$, $G_2\square P_n$, and $G_3\square P_n$ is at least $n-1$. The graphs $G_4\square P_n$, $G_5\square P_n$, $G_6\square P_n$, and $G_7\square P_n$ contain the graph $S_4\square P_n$ as a subgraph. As $\mathrm{cr}(S_4\square P_n) = 2(n-1)$ (see [6]), $\mathrm{cr}(G_4\square P_n) \geq 2(n-1)$, $\mathrm{cr}(G_5\square P_n) \geq 2(n-1)$, $\mathrm{cr}(G_6\square P_n) \geq 2(n-1)$, and also $\mathrm{cr}(G_7\square P_n) \geq 2(n-1)$. The graph $S_5\square P_n$ is a subgraph of the graphs $G_8\square P_n$, $G_9\square P_n$, $G_{10}\square P_n$, and $G_{11}\square P_n$. As $\mathrm{cr}(S_5\square P_n) = 4(n-1)$ (see [1]), we have lower bound for the crossing numbers of these graphs. The graphs $G_{12}\square P_n$ and $G_{13}\square P_n$ contain the graph $S_6\square P_n$ as a subgraph and the graph $G_{14}\square P_n$ contains the graph $S_7\square P_n$ as a subgraph. It was proved in [1], that $\mathrm{cr}(S_6\square P_n) = 6(n-1)$, and $\mathrm{cr}(S_7\square P_n) = 9(n-1)$. So, the lower bounds $6(n-1)$, $6(n-1)$, and $9(n-1)$ are for the crossing numbers of the graphs $G_{12}\square P_n$, $G_{13}\square P_n$, and $G_{14}\square P_n$, respectively.

The upper and lower bounds of crossing numbers for graphs $G_i\square P_n$ for $i = 1, \ldots, 14$ are the same. Thus, we get exact values of crossing numbers of corresponding graphs. □

Theorem 2. *For* $n \geq 1$, $\mathrm{cr}(G_{15}\square P_n) = \mathrm{cr}(G_{16}\square P_n) = \mathrm{cr}(G_{17}\square P_n) = \mathrm{cr}(G_{18}\square P_n) =$

$$= \mathrm{cr}(G_{19}\square P_n) = \mathrm{cr}(G_{20}\square P_n) = 2(n-1).$$

Proof: The drawings of the graphs $G_{15}\square P_n$, $G_{16}\square P_n$, $G_{17}\square P_n$, $G_{18}\square P_n$, $G_{19}\square P_n$, and $G_{20}\square P_n$ with $2(n-1)$ crossings are shown in Figures 3(a) – (f). Thus, $cr(G_i\square P_n) \leq 2(n-1)$ for $i = 15\ldots, 20$. The graphs G_{15}, G_{16}, and G_{17} contain the graph T (see Figure 5) as a subgraph and the graphs G_{18}, G_{19}, and G_{20} contain the subdivision of the graph T (subdivision of the edge which is incident to the vertices of degree three). So, the graph $T\square P_n$ or its subdivision is a subgraph of the graphs $G_i\square P_n$. It was published in [7], that $\mathrm{cr}(T\square P_n) = 2(n-1)$. Hence, the crossing numbers of graphs $G_i\square P_n$ for $i = 16, \ldots, 21$ are at least $2(n-1)$. Thus, $\mathrm{cr}(G_{15}\square P_n) = \mathrm{cr}(G_{16}\square P_n) = \mathrm{cr}(G_{17}\square P_n) = \mathrm{cr}(G_{18}\square P_n) =$

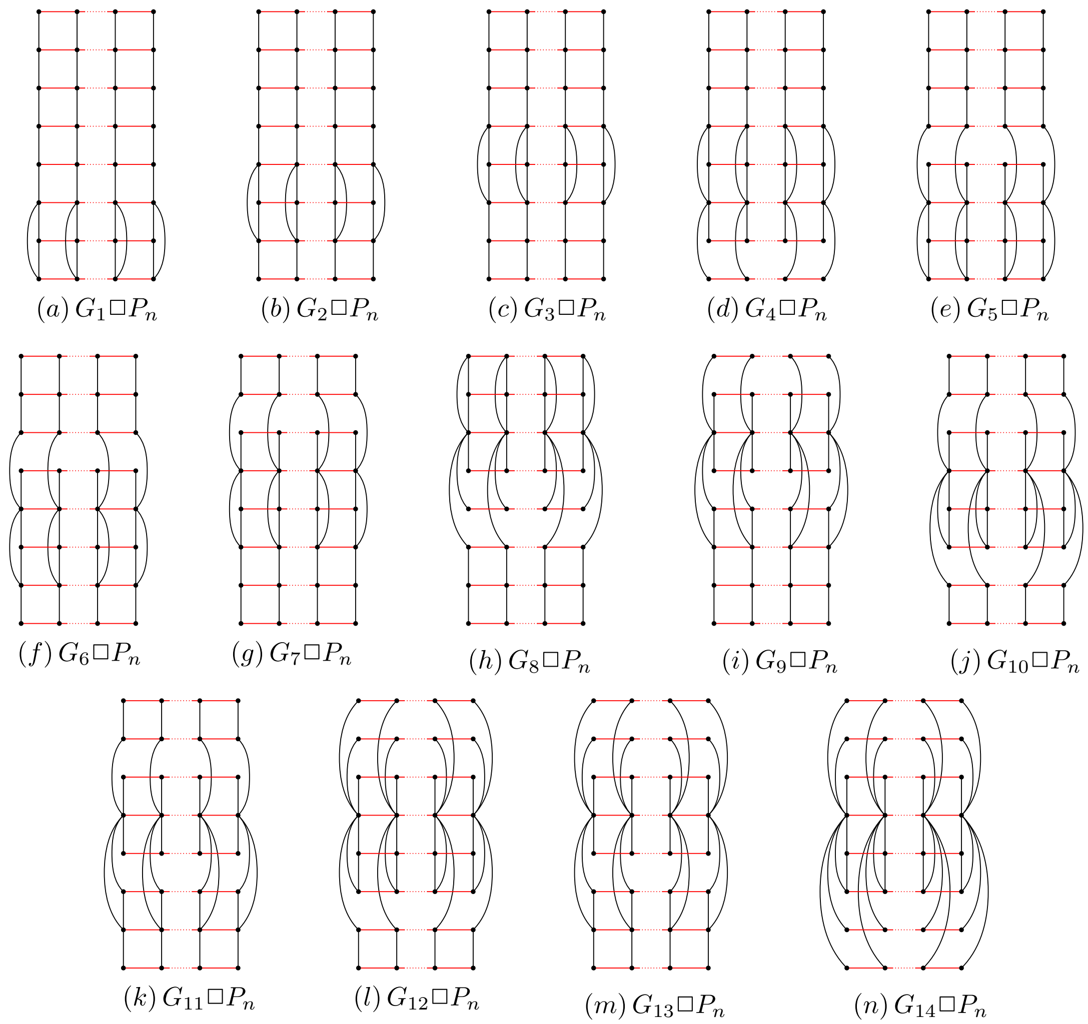

Figure 2: Cartesian products $G_i\Box P_n$ for $i = 1, \ldots, 14$

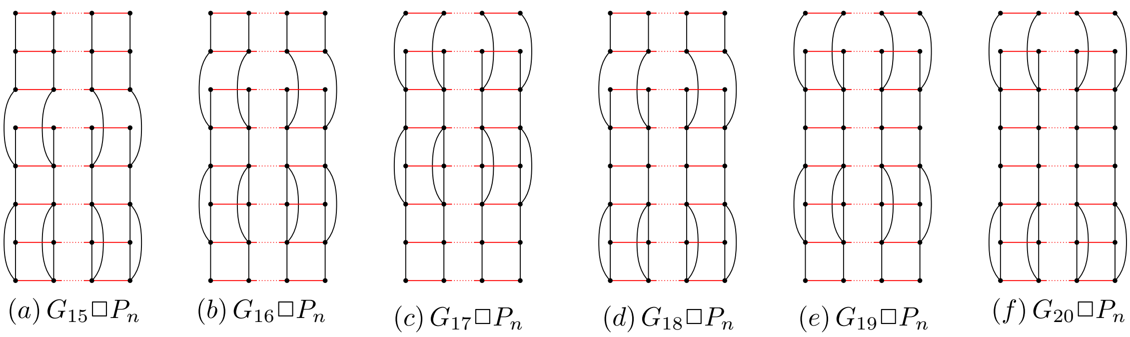

Figure 3: Cartesian products $G_i\Box P_n$ for $i = 15, \ldots, 20$

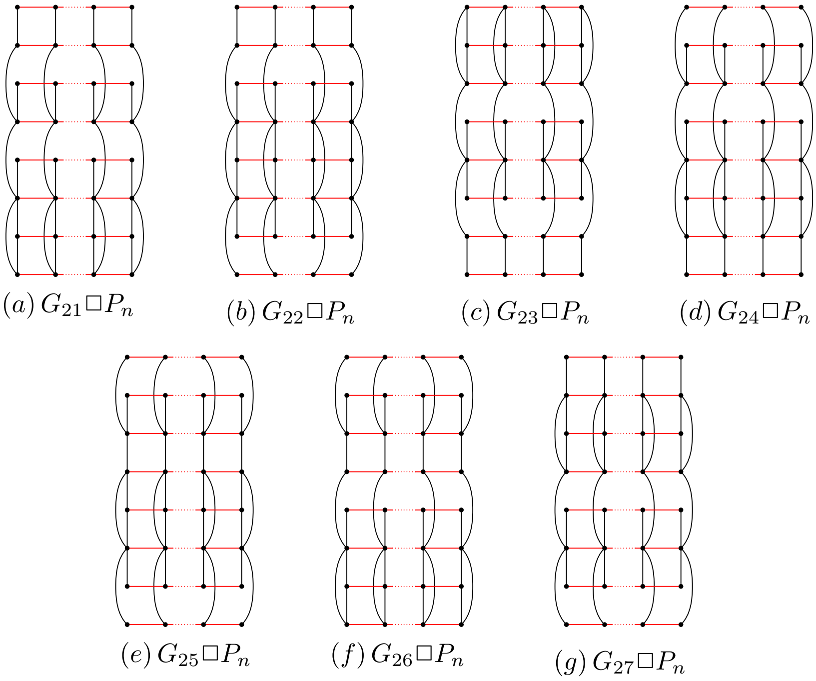

Figure 4: Cartesian products $G_i\square P_n$ for $i = 21, \ldots, 27$

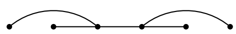

Figure 5: The graph T

$= \mathrm{cr}(G_{19}\square P_n) = \mathrm{cr}(G_{20}\square P_n) = 2(n-1).$ □

Theorem 3. *For $n \geq 1$,* $\mathrm{cr}(G_{21}\square P_n) = \mathrm{cr}(G_{22}\square P_n) = \mathrm{cr}(G_{23}\square P_n) = \mathrm{cr}(G_{24}\square P_n) =$
$= \mathrm{cr}(G_{25}\square P_n) = \mathrm{cr}(G_{26}\square P_n) = \mathrm{cr}(G_{27}\square P_n) = 3(n-1).$

Proof: There are drawings of the graphs $G_{21}\square P_n$, $G_{22}\square P_n$, $G_{23}\square P_n$, $G_{24}\square P_n$, $G_{25}\square P_n$, $G_{26}\square P_n$, and $G_{27}\square P_n$ (see Figure 4(a) − (g)) with $3(n-1)$ crossings. Thus, the crossing number for every graph $G_i\square P_n$, $i = 21, \ldots, 27$ is at most $3(n-1)$. The graph S (see Figure 6) is a subgraph of every graph G_i for $i = 21, \ldots, 27$. So, the graph $S\square P_n$ is a subgraph of the graphs $G_i\square P_n$. As $\mathrm{cr}(S\square P_n) = 3(n-1)$ (see [2]), $\mathrm{cr}(G_i\square P_n) \geq 3(n-1)$ for every $i = 21, \ldots, 27$. This completes our proof. □

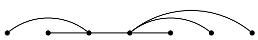

Figure 6: The graph S

References

[1] BOKAL, D.: On the crossing numbers of Cartesian products with paths, *J. Graph Theory B* **97** (2007), 381–384.

[2] DRAŽENSKÁ, E.: The crossing numbers of products of paths with 7-vertex trees, *Creative Mathematics and Informatics*, **23** (2014), 109–119.

[3] GAREY, M. R., JOHNSON, D. S.: Crossing number is NP-complete, *SIAM J. Algebraic and Discrete Methods* **4**, (1983) 312–316.

[4] JENDROL', S., ŠČERBOVÁ,M.: On the crossing numbers of $S_m \times P_n$ and $S_m \times C_n$, *Časopis pro pěstování matematiky* **107** (1982), 225–230.

[5] KLEITMAN, D. J.: The crossing number of $K_{5,n}$ *J. Comb. Theory* **9**, (1970) 315–323.

[6] KLEŠČ, M.: On the crossing numbers of Cartesian products of stars and paths or cycles, *Mathematica Slovaca* **41** (1991), 113–120.

[7] KLEŠČ, M., PETRILLOVÁ, J.:The crossing numbers of products of paths with graphs of order six, *Discussiones Mathematicae - Graph theory* **33** (2013), 571–582.

[8] TURÁN, P.: "A note of welcome", *J. Graph Theory* **1**, (1977) 7–9.

[9] WOODALL, D.R.: Cyclic–order graphs and Zarankiewicz's crossing–number conjecture, *J. Graph Theory* **17** (1993), 657–671.

[10] ZARANKIEWICZ, K.: "On a problem of P. Turán concerning graph", *Fund. Math.* **41**, (1977) 137–145.