http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 7, No 2, pp. 15 – 24 (2019) doi:10.26456/mmg/2019-722

Partial differential equations: From zero integrals to exact solutions and Bäcklund transforms

Nikolay L. Chuprikov

Department of Theoretical Physics, Tomsk State Pedagogical University, 634041, Tomsk, Russia
e-mail: chnl@tspu.edu.ru

Received 2 August 2019, in final form 25 August. Published 26 August 2019.

Abstract. Solving of nonlinear partial differential equations (PDEs) for functions of two independent variables is reduced to solving of a system of two linear PDEs of the first order for zero integrals which implicitly define unknown functions and their derivatives entering these equations. Using the example of the S-Gordon equation, it is shown that this approach is universal in the sense that it can be equally used both to solve analytically the original PDEs and to find Bäcklund transforms.

Keywords: zero integrals, nonlinear partial differential equations, Bäcklund transforms
MSC numbers: 35A20, 35A30

1. Introduction

As is known, the basic equations of motion in physics are formulated in the form of differential equations for nontrivial (nonconstant) functions of one or more variables. These are either ordinary differential equations (ODEs) or partial differential equations (PDEs). For example, in the Hamiltonian description of classical particle dynamics, we are dealing with the ODEs for functions that describe the change in time of the position and momentum of a particle. Quantum particle dynamics is described by the PDE formulated for the wave function in the coordinate representation, which depends on the time and particle coordinate. Thus, for physicists dealing with ODEs and PDEs, it is very important to obtain their solutions in the form of functions that are determined (explicitly or implicitly) by exact analytical expressions. And, as is known, at present the most effective methods developed to solve this problem are symmetry methods (see, for example, [1, 2]), the main advantage of which is that they can in principle be applied to arbitrary ODEs and PDEs.

However, not every differential equation has any symmetry. In this regard, the question arises: is there a way to increase the chance to successfully find exact analytical solutions of investigated differential equations, before applying symmetry methods to them? In our opinion, the answer is in the affirmative, and our arguments in favor of the existence of such a way are as follows. It is a priori obvious that it is impossible to find a common solution to the original ODE in the form of an explicit function, represented by an exact analytic expression, when its common solution is in reality an implicit or parametric function. In this connection, the idea arises that the initial differential equation, before its solving, should be transformed into equation(s) for functions from an optimally wide class to contain explicit, implicit, and parametric functions as particular cases. We propose to use for this purpose a class of functions that are implicitly defined, together with the derivatives included in the original differential equation, by the system of zero integrals of this equation.

This idea was first proposed and implemented in [3], by the example of a system of ODEs of general form, which is unsolvable with respect to higher derivatives. The main goal of the present paper is to show that idea is also applicable to PDEs of general form. The effectiveness of this approach is demonstrated in Section 2 by the example of the S-Gordon equation, where the sine-Gordon equation is considered in detail. In addition, in Section 3 it is shown that this approach is useful also for finding the Bäcklund transforms.

2. Zero integrals of the S-Gordon equation

Let us consider the S-Gordon equation for the unknown function $u = u(x_1, x_2)$:

$$\Phi\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}\right) \equiv \frac{\partial^2 u}{\partial x_1 \partial x_2} - S(u) = 0, \tag{1}$$

where $S(u)$ is a given function. We will assume that the very function u and its derivatives $u_1 = \frac{\partial u}{\partial x_1}$, $u_2 = \frac{\partial u}{\partial x_2}$, $u_{12} = \frac{\partial^2 u}{\partial x_1 \partial x_2}$ are implicit functions given by zero integrals of this equation. That is, in the six-dimensional $\tilde{U}$-space, where $\tilde{U} = (x_1, x_2, u, u_1, u_2, u_{12})$, we have the system of transcendental or algebraic equations

$$\begin{aligned} \Phi\left(x_1, x_2, u, u_1, u_2, u_{12}\right) &\equiv u_{12} - S(u) = 0, \\ \tilde{F}_n\left(x_1, x_2, u, u_1, u_2, u_{12}\right) &= c_n, \quad n = 1, \ldots \end{aligned} \tag{2}$$

(we do not fix the number of zero integrals).

On the surface $\Phi(\tilde{U}) = 0$ these equations can be reduced to the equations $F_n(U) = c_n, \ \ n = 1, \ldots$ where $U = (x_1, x_2, u, u_1, u_2)$. In the U-space, all zero integrals must obey the same system of equations which can written (omitting the index 'n') in the form

$$\hat{D}_1 F(U) = 0, \qquad \hat{D}_2 F(U) = 0 \tag{3}$$

where $\hat{D}_1$ and $\hat{D}_2$ are the operators of total derivatives with respect to x_1 and x_2, respectively:

$$\hat{D}_1 = \frac{\partial}{\partial x_1} + u_1 \frac{\partial}{\partial u} + L_{11}(U) \frac{\partial}{\partial u_1} + S(u) \frac{\partial}{\partial u_2}, \quad \hat{D}_2 = \frac{\partial}{\partial x_2} + u_2 \frac{\partial}{\partial u} + S(u) \frac{\partial}{\partial u_1} + L_{22}(U) \frac{\partial}{\partial u_2}, \tag{4}$$

where $L_{11}(U) \equiv \partial u_1 / \partial x_1$ and $L_{22}(U) \equiv \partial u_2 / \partial x_2$ are unknown functions in the U-space, which must obey the condition $(\hat{D}_1 \hat{D}_2 - \hat{D}_2 \hat{D}_1) F(U) = 0$ – in this case the system (3) of linear PDEs is Jacobian.

Direct calculations show that this condition is reduced to the equations

$$\begin{aligned} \frac{\partial L_{22}}{\partial x_1} + u_1 \frac{\partial L_{22}}{\partial u} + L_{11} \frac{\partial L_{22}}{\partial u_1} + S(u) \frac{\partial L_{22}}{\partial u_2} - u_2 \frac{dS}{du} &= 0, \\ \frac{\partial L_{11}}{\partial x_2} + u_2 \frac{\partial L_{11}}{\partial u} + S(u) \frac{\partial L_{11}}{\partial u_1} + L_{22} \frac{\partial L_{11}}{\partial u_2} - u_1 \frac{dS}{du} &= 0 \end{aligned} \tag{5}$$

which can also be rewritten in the form

$$\hat{D}_1 L_{22}(U) = u_2 \frac{dS(u)}{du}, \qquad \hat{D}_2 L_{11}(U) = u_1 \frac{dS(u)}{du}.$$

As is seen, Eqs. (5) are nonlinear. Thus, within the framework of the technique of zero integrals, the main difficulty is to find such functions $L_{11}(U)$ and $L_{22}(U)$ when Eqs. (3) represent Jacobian system.

An attempt to solve Eqs. (5) for an arbitrary function $S(u)$, with making use of the Maple package, turns out to be successful only for $S(u) = e^{au}$, where a is a real constant (this is related to the fact that the S-Gordon equation with $S(u) = e^{au}$ is distinguished in solving the problem of finding Bäcklund transforms for the S-Gordon equation). For example, if we assume that, for $S(u) = e^{au}$, both functions in (5) do not depend on x_1 and x_2, then direct calculation with Maple gives $L_{11} = \frac{au_1^2}{2} + C$ and

$$L_{22} = 2au_1 e^{au} \frac{(au_1^2+2C)u_2 - u_1 e^{au}}{(au_1^2+2C)^2} + W\left(\frac{(au_1^2+2C)e^{-au}}{a}, u_2 - \frac{2u_1 e^{au}}{au_1^2+2C}\right);$$

here W is an arbitrary function of two variables, and C is an arbitrary constant.

At the same time, there is an easy way to find a particular solution to Eqs. (5) for an arbitrary function $S(u)$. To do this, it is enough to equate the ratio of the coefficients facing the derivatives $\partial F/\partial u_2$ and $\partial F/\partial u$, of the first equation in (3), to the ratio of similar coefficients of the second equation. This results in the algebraic equation on the functions $L_{11}(U)$ and $L_{22}(U)$. The second equation for these functions can be obtained if one performs the analogous procedure with the coefficients before the derivatives $\partial F/\partial u_1$ and $\partial F/\partial u$. As a result, we obtain

$$\frac{L_{11}(U)}{u_1} = \frac{S(u)}{u_2}, \qquad \frac{S(u)}{u_1} = \frac{L_{22}(U)}{u_2}.$$

It is easy to check that the functions

$$L_{11}(U) = \frac{u_1}{u_2}\, S(u), \qquad L_{22}(U) = \frac{u_2}{u_1}\, S(u) \tag{6}$$

satisfy Eqs. (5).

For example, with these functions $L_{11}(U)$ and $L_{22}(U)$, Eqs. (3) for $S = \sin(u)$ take the form

$$\frac{\partial F}{\partial x_1} + u_1\frac{\partial F}{\partial u} + \frac{u_1}{u_2}\sin(u)\frac{\partial F}{\partial u_1} + \sin(u)\frac{\partial F}{\partial u_2} = 0;\ \frac{\partial F}{\partial x_2} + u_2\frac{\partial F}{\partial u} + \sin(u)\frac{\partial F}{\partial u_1} + \frac{u_2}{u_1}\sin(u)\frac{\partial F}{\partial u_2} = 0 \tag{7}$$

Their solution, with the help of the Maple package, can be obtained as follows. We first solve these equations separately and find two common integrals:

$$F_1(U) \equiv \frac{u_2}{u_1} = \alpha, \quad F_2(U) \equiv u_1 u_2 + 2\cos(u) = \beta$$

where α and β are arbitrary constants. Next, we make a change of variables $u_1 = \sqrt{\alpha(\beta - 2\cos(u))}/\alpha$ and $u_2 = \sqrt{\alpha(\beta - 2\cos(u))}$, and reduce Eqs. (7) to the form

$$\frac{\partial F}{\partial x_1} + \sqrt{\frac{\beta - 2\cos(u)}{\alpha}}\,\frac{\partial F}{\partial u} = 0; \quad \frac{\partial F}{\partial x_2} + \sqrt{\alpha[\beta - 2\cos(u)]}\,\frac{\partial F}{\partial u} = 0.$$

The solution of this system is the integral

$$F_3(U) \equiv x_2 + \frac{x_1}{\alpha} \pm \frac{2}{\sqrt{\alpha(\beta+2)}}\,\mathrm{EllipticF}\left[\cos\left(\frac{u}{2}\right), \frac{2}{\sqrt{\beta+2}}\right].$$

Further, by solving the equation $F_3(U) = \phi$ with respect to u, where ϕ is an arbitrary constant, we finally obtain a three-parameter family of solutions of the sine-Gordon equation

$$u_{(\pm)}(x_1, x_2) = 2 \arccos \left\{ \pm \text{JacobiSN} \left[\frac{\sqrt{\beta+2}}{2} \left(\frac{x_1}{\sqrt{\alpha}} + \sqrt{\alpha}(x_2 - \phi) \right), \frac{2}{\sqrt{\beta+2}} \right] \right\}. \quad (8)$$

For $\alpha > 0$ and $\beta \geq -2$ these solutions are real. In this case, $u_{(\pm)}(x_1, x_2) \to \pi$ in the limit $\beta \to -2$. For $-2 < \beta < 2$ these solutions describe nonlinear waves, and for $\beta = 2$ we have kink and antikink solutions.

As is known, the symmetry of the sine-Gordon equation is such that if $u(x_1, x_2)$ is a solution, then the functions $2\pi n + u(x_1, x_2)$ and $2\pi n - u(x_1, x_2)$, where n is an arbitrary integer, are also solutions. That is, any particular solution of this equation is, strictly speaking, an infinite-valued function. As for the expressions (8), each of them gives that branch of the corresponding infinite-valued solution, the change domain of which is the interval $[0, 2\pi]$. Wherein it is important to emphasize that for $-2 < \beta < 2$, each of the two functions $u_{(\pm)}(x_1, x_2)$ is smooth with respect to each argument and describes a nonlinear wave, the graph of which is shifted to the left with increasing the second argument. If $\beta > 2$, then each of these two functions gives, in the interval $[0, 2\pi]$, a non-smooth representation of the corresponding infinite-valued function, each the branch of which is a smooth monotonically increasing or decreasing function that varies in the interval $(-\infty, \infty)$.

Our next step is to show that this approach is useful not only for searching for analytical solutions of nonlinear PDEs, but also for finding the Bäcklund transforms.

3. Bäcklund Transforms for the S-Gordon equation

The above idea is fully applicable to finding Bäcklund transforms for PDEs (if any). Again, we demonstrate this by the example of the S-Gordon equation. For this purpose let us consider two S-Gordon equations (our approach differs from the one used in [4] for finding Bäcklund transforms in this case):

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = S_u(u), \qquad \frac{\partial^2 v}{\partial x_1 \partial x_2} = S_v(v), \quad (9)$$

where $S_u(u)$ and $S_v(v)$ are given functions (these can be the same); $u(x_1, x_2)$ and $v(x_1, x_2)$ are searched-for functions.

Again, in line with our approach let us proceed to the extended space in which the unknown functions and their derivatives play the role of independent variables. A little modification is that now we take $U = (p, m, m_1, p_2)$; here $p = u + v$, $m = u - v$, $m_1 = u_1 - v_1$, $p_2 = u_2 + v_2$; $u_1 = \partial u/\partial x_1$, $u_2 = \partial u/\partial x_2$, $v_1 = \partial v/\partial x_1$, $v_2 = \partial v/\partial x_2$. To avoid undesirable unwieldiness, we do not include the variables $m_2 = u_2 - v_2$ and $p_1 = u_1 + v_1$ in this list, since this approach would still required to exclude them.

Differentiating the zero integral $F(U)$ of Eqs. (9) with respect to x_1 and x_2 gives

$$\left(\frac{\partial F}{\partial p}+\frac{\partial F}{\partial m}\right)u_1+\frac{\partial F}{\partial m_1}L_{u_{11}}+\frac{\partial F}{\partial p_2}S_u(u)+\left(\frac{\partial F}{\partial p}-\frac{\partial F}{\partial m}\right)v_1-\frac{\partial F}{\partial m_1}L_{v_{11}}-\frac{\partial F}{\partial p_2}S_v(v)=0,$$
$$\left(\frac{\partial F}{\partial p}+\frac{\partial F}{\partial m}\right)u_2+\frac{\partial F}{\partial m_1}S_u(u)+\frac{\partial F}{\partial p_2}L_{u_{22}}+\left(\frac{\partial F}{\partial p}-\frac{\partial F}{\partial m}\right)v_2-\frac{\partial F}{\partial m_1}S_v(v)-\frac{\partial F}{\partial p_2}L_{v_{22}}=0,$$

where $L_{u_{11}}(U)$, $L_{v_{11}}(U)$, $L_{u_{22}}(U)$ and $L_{v_{22}}(U)$ are unknown functions. Passing everywhere to the auxiliary variables p, m, m_1 and p_2, we obtain a system of two equations for the function $F(U)$:

$$\hat{D}_1F\equiv p_1\frac{\partial F}{\partial p}+m_1\frac{\partial F}{\partial m}+f_1(U)\frac{\partial F}{\partial m_1}+\left[S_u\left(\frac{p+m}{2}\right)+S_v\left(\frac{p-m}{2}\right)\right]\frac{\partial F}{\partial p_2}=0,$$
$$\hat{D}_2F\equiv p_2\frac{\partial F}{\partial p}+m_2\frac{\partial F}{\partial m}+\left[S_u\left(\frac{p+m}{2}\right)-S_v\left(\frac{p-m}{2}\right)\right]\frac{\partial F}{\partial m_1}+f_2(U)\frac{\partial F}{\partial p_2}=0.\quad(10)$$

where $f_1(U)=L_{u_{11}}(U)-L_{v_{11}}(U)$, $f_2(U)=L_{u_{22}}(U)+L_{v_{22}}(U)$.

The system of Eqs. (10) is Jacobian, provided that $(\hat{D}_1\hat{D}_2-\hat{D}_2\hat{D}_1)F=0$. To solve this equation, we will proceed in the same way as in the previous section. Let us equate the ratio of the coefficients before the derivatives $\partial F/\partial m_1$ and $\partial F/\partial p$ in the first equation (3) to the ratio of analogous coefficients in the second equation. Besides, let the ratio of the coefficients before derivatives $\partial F/\partial p_2$ and $\partial F/\partial m$ in the first equation (3) be equal to the ratio of analogous coefficients in the second equation. This gives two algebraic equations

$$\frac{f_1(U)}{p_1}=\frac{S_u\left(\frac{p+m}{2}\right)-S_v\left(\frac{p-m}{2}\right)}{p_2},\qquad\frac{S_u\left(\frac{p+m}{2}\right)+S_v\left(\frac{p-m}{2}\right)}{m_1}=\frac{f_2(U)}{m_2}.$$

From here it follows that

$$f_1(U)=\frac{p_1}{p_2}\left[S_u\left(\frac{p+m}{2}\right)-S_v\left(\frac{p-m}{2}\right)\right],\qquad f_2(U)=\frac{m_2}{m_1}\left[S_u\left(\frac{p+m}{2}\right)+S_v\left(\frac{p-m}{2}\right)\right].\quad(11)$$

It is not difficult to show that Eqs. (10) can be reduced in this case to the form

$$\hat{X}_1F\equiv\frac{\partial F}{\partial m}+\frac{1}{m_1}\left[S_u\left(\frac{p+m}{2}\right)+S_v\left(\frac{p-m}{2}\right)\right]\frac{\partial F}{\partial p_2}=0,$$
$$\hat{X}_2F\equiv\frac{\partial F}{\partial p}+\frac{1}{p_2}\left[S_u\left(\frac{p+m}{2}\right)-S_v\left(\frac{p-m}{2}\right)\right]\frac{\partial F}{\partial m_1}=0.\qquad(12)$$

One can show that $\hat{X}_1(\hat{X}_2F)-\hat{X}_2(\hat{X}_1F)$ is proportional to the product $J(U)\left(p_2\frac{\partial F}{\partial p_2}-m_1\frac{\partial F}{\partial m_1}\right)$, where

$$J(U)=2S_u^2\left(\frac{p+m}{2}\right)-2S_v^2\left(\frac{p-m}{2}\right)-m_1p_2S_u'\left(\frac{p+m}{2}\right)-m_1p_2S_v'\left(\frac{p-m}{2}\right);\quad(13)$$

here the prime denotes the derivative. Thus, Eqs. (12) constitute a Jacobian system if $J(U)=0$ (otherwise it would have to be supplemented by the equation

$$p_2\frac{\partial F}{\partial p_2}-m_1\frac{\partial F}{\partial m_1}=0;$$

we will exclude this variant).

The equation $J(U)=0$ implies that the function $J(U)$ is an integral of Eqs. (12). We substitute it into Eqs. (12) and get two equations. In the original variables they are

$$\begin{aligned}2\left[S_u(u)-S_v(v)\right]\left[S_u'(u)-S_v'(v)\right]-(u_1-v_1)(u_2+v_2)\left[S_u''(u)-S_v''(v)\right]=0;\\ 2\left[S_u(u)+S_v(v)\right]\left[S_u'(u)-S_v'(v)\right]-(u_1-v_1)(u_2+v_2)\left[S_u''(u)+S_v''(v)\right]=0.\end{aligned}\quad(14)$$

The equation $J(U)=0$, in the original variables, has the form

$$2S_u(u)^2-2S_v(v)^2-(u_1-v_1)(u_2+v_2)\left[S_u'(u)+S_v'(v)\right]=0.\quad(15)$$

We find from it the quantity $(u_1-v_1)(u_2+v_2)$ and reduce the equations (14) to the form

$$\begin{aligned}&[S_u(u)+S_v(v)][S_u''(u)-S_v''(v)]-[S_u'(u)]^2+[S_v'(v)]^2=0;\\ &[S_u(u)-S_v(v)]\left[S_u''(u)+S_v''(v)\right]-[S_u'(u)]^2+[S_v'(v)]^2=0.\end{aligned}\quad(16)$$

Subtracting and summing these two equations, and then separating the variables u and v, we obtain

$$\frac{S_u''(u)}{S_u(u)}=\frac{S_v''(v)}{S_v(v)}=k;\quad [S_u'(u)]^2-S_u(u)S_u''(u)=[S_v'(v)]^2-S_v(v)S_v''(v)=a,\quad(17)$$

where k and a are arbitrary real constants.

In solving these two equations, it is necessary to distinguish the general case, when both functions $S_u(u)$ and $S_v(v)$ are nonzero, and the particular case, when one of these functions (for example, $S_v(v)$) is identically equal to zero. We start with the general case.

The general case, in its turn, is subdivided into the following three variants: (a) $k\neq 0$; (b) $k=0$ but $a\neq 0$; (c) $k=a=0$. Let's consider each of them separately.

Variant (a). The solution of Eqs. (17) for $k\neq 0$ shows that the Bäcklund transform between the equations $u_{12}=S_u(u)$ and $v_{12}=S_v(v)$ exists if their right-hand sides are defined by one of two expressions

$$S_{(+)}(u)=\frac{e^{(u-C)\sqrt{k}}-ae^{-(u-C)\sqrt{k}}}{2\sqrt{k}};\qquad S_{(-)}(u)=-\frac{ae^{(u-C)\sqrt{k}}-e^{-(u-C)\sqrt{k}}}{2\sqrt{k}},$$

where C is an arbitrary constant (unlike the constants k and a it may have different values for S_u and S_v). In particular,

- $S_{(\pm)} = \pm\sin(u - C)$, for $k = -1$, $a = 1$;
- $S_{(\pm)} = \pm\sinh(u - C)$, for $k = 1$, $a = 1$;
- $S_{(\pm)} = \frac{1}{2}e^{\pm(u-C)}$, for $k = 1$, $a = 0$.

Further, with restricting ourselves to the solution $S_{(+)}(u)$, we assume that

$$S_u(u) = \frac{e^{(u-C_u)\sqrt{k}} - ae^{-(u-C_u)\sqrt{k}}}{2\sqrt{k}}, \qquad S_v(v) = \frac{e^{(v-C_v)\sqrt{k}} - ae^{-(v-C_v)\sqrt{k}}}{2\sqrt{k}}. \tag{18}$$

Note that Eqs. (12) with these functions have, in addition to the integral $J(U)$, one more integral –

$$\frac{m_1}{e^{\sqrt{k}p/2} - ae^{\sqrt{k}(C_u+C_v-p/2)}} = \alpha, \tag{19}$$

where α is an arbitrary constant.

As a result, from the equations $J(U) = 0$ and (19) it follows that

$$m_1 = \alpha\left(e^{\sqrt{k}p/2} - ae^{\sqrt{k}(C_u+C_v-p/2)}\right); \quad p_2 = \frac{e^{\sqrt{k}(m/2-C_u)} - e^{-\sqrt{k}(m/2+C_v)}}{k\,\alpha}. \tag{20}$$

With $p = u + v$, $m = u - v$, $p_2 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_2}$ and $m_1 = \frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_1}$, the equalities (20) represent searched-for Bäcklund transforms acting between the S-Gordon equations with the right parts (18).

Note that the most interesting are particular cases of the transforms (20), which correspond to $C_u = C_v = 0$ (this is due to the fact that the equation $u_{x_1x_2} = S(u - C)$, with help of replacement $\tilde{u} = u - C$, is reduced to the equation $\tilde{u}_{x_1x_2} = S(\tilde{u})$). These cases are as follows:

- for $a = 1$, $k \to -k^2$, $\alpha \to -i\alpha$

$$u_1 - v_1 = 2\alpha\sin\left(k\frac{u+v}{2}\right), \quad u_2 + v_2 = \frac{2}{\alpha k^2}\sin\left(k\frac{u-v}{2}\right); \quad S_u(u) = S_v(u) = \frac{\sin(ku)}{k};$$

- for $a = 1$, $k \to k^2$

$$u_1 - v_1 = 2\alpha\sinh\left(k\frac{u+v}{2}\right), \quad u_2 + v_2 = \frac{2}{\alpha k^2}\sinh\left(k\frac{u-v}{2}\right); \quad S_u(u) = S_v(u) = \frac{\sinh(ku)}{k};$$

- for $a = 0$, $k \to k^2$

$$u_1 - v_1 = \alpha\exp\left(k\frac{u+v}{2}\right), \quad u_2 + v_2 = \frac{2}{\alpha k^2}\sinh\left(k\frac{u-v}{2}\right); \quad S_u(u) = S_v(u) = \frac{e^{ku}}{2k}.$$

Variant (b). Note that the integral (19) has a singularity as $k \to 0$. In this case a non-trivial solution exists only for $a = 1$. In the limit $k \to 0$ for $a = 1$, instead of the integral (19), we have

$$\frac{m_1}{p - C_u - C_v} = \alpha. \tag{21}$$

Further, as in the previous variant, with making use of this equation we exclude m_1 from the equation $J(U) = 0$ and obtain, instead of (20), Bäcklund transforms for $S_u(u) = u - C_u$ and $S_v(v) = v - C_v$:

$$u_1 - v_1 = \alpha(u + v - C_u - C_v); \quad u_2 + v_2 = \frac{u - v - C_u + C_v}{\alpha}. \tag{22}$$

Variant (c). For $a = 0$ and $k \to 0$, the solution of Eqs. (17) are functions $S_u(u) = C_u$ and $S_v(v) = C_v$, now $J(U) = 2(C_v^2 - C_u^2)$. Thus, Eqs. (12) constitute a Jacobian system if $C_v = \pm C_u$.

For example, for $C_v = C_u$ two independent integrals of Eqs. (12) can be written in the form $m_1 = \alpha$ and $p_2 - 2C_u m/m_1 = \beta$, and the corresponding Bäcklund transforms are

$$u_1 - v_1 = \alpha; \qquad u_2 + v_2 = \frac{2C_u}{\alpha}(u - v) + \beta. \tag{23}$$

If $C_v = -C_u$, then the two independent integrals of Eqs. (12) are $p_2 = \beta$ and $m_1 - 2C_u p/p_2 = \alpha$, and the corresponding Bäcklund transforms are

$$u_1 - v_1 = \frac{2C_u}{\beta}(u + v) + \alpha; \qquad u_2 + v_2 = \beta. \tag{24}$$

In **the particular case**, when $S_u(u) \neq 0$ while $S_v(v) \equiv 0$, the function $S_u(u)$ obeys the equation

$$[S_u'(u)]^2 - S_u(u) S_u''(u) = 0, \tag{25}$$

from which it follows that $S_u(u) = c\, e^{ku}$; here c and k are arbitrary real constants.

In this case the two independent integrals of the equations (12) – the already known integral $J(U)$ (see (13)) and the new integral $J_1(U)$ – can be written in the form

$$J(U) = -kp_2 m_1 + 2c\, e^{k(p+m)/2} = 0, \qquad J_1(U) = m_1 e^{-kp/2} = \alpha;$$

where α is an arbitrary real constant.

From here we find the Bäcklund transforms for $S_u(u) = c\, e^{ku}$ and $S_v(u) = 0$:

$$u_1 - v_1 = \alpha \exp\left(k\frac{u+v}{2}\right), \qquad u_2 - v_2 = \frac{2c}{\alpha k} \exp\left(k\frac{u-v}{2}\right). \tag{26}$$

4. Conclusion

By the example of the S-Gordon equation it is shown that solving of nonlinear PDEs is reduced to solving the system of linear PDEs of the first order for zero integrals of the original PDEs, which sets searched-for functions and their derivatives as implicit functions. As is shown, this system contains unknown coefficients, and the main difficulty in this approach is to find such coefficients for which the system of PDEs for zero integrals is Jacobian. Making use of symmetry methods in solving this task could expand the class of solutions of nonlinear PDEs in analytical form. Of course, PDEs which arise for such coefficients in the case of the S-Gordon equation (see Eqs. (5)) are set in the five-dimensional U-space, rather than in the two-dimensional (x_1, x_2)-space. But, at the same time, the type of these equations is fixed. It does not depend on the function $S(u)$. Moreover, analogous equations arise in searching for zero integrals of any original PDE of the second order for a function of two independent variables.

References

[1] J. Fuchs, C. Schweigert, Symmetries, Lie Algebras and Representations. Cambridge University Press 1997, pp. 438

[2] P. E. Hydon, Symmetry Methods for Differential Equations. Cambridge University Press. 2000, pp.213

[3] Chuprikov N.L. A new approach to finding general and singular solutions of certain classes of differential equations. Deposited in VINITI, 1987, No. 3219-B87 (in Russian); see also its version doi: 10.13140/RG.2.2.28369.22888

[4] Shadwick W. F. The Backlund problem for the equation $\partial^2 z/\partial x_1 \partial x_2 = f(z)$. J. of Math. Phys. **19**, 2312 (1978) doi: 10.1063/1.523611

[5] David W. McLaughlin, and Alwyn C. Scott, A restricted Bäcklund transformation. Journal of Mathematical Physics **14**, 1817 (1973) doi: 10.1063/1.1666254