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Alternative proof on the crossing number of $K_{2,3,n}$

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Abstract. The main aim of the paper is to give the crossing number of join product $G + D_n$ for the connected graph G of order five isomorphic with the complete bipartite graph $K_{2,3}$, where D_n consists on n isolated vertices. The proof of the crossing number of $K_{2,3,n}$ was published by a partially unclear discussion of cases by Asano in [1]. In our proof, it will be used an idea of cyclic permutations and their combinatorial properties. Finally, by adding one edge to the graph G, we are able to obtain the crossing number of the join product with the discrete graph D_n for one new graph.

Keywords: graph, drawing, crossing number, join product, cyclic permutation

MSC numbers: 05C10, 05C38

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1. Introduction

The investigation on the crossing number of graphs is a classical and very difficult problem provided that an computing of the crossing number of a given graph in general is NP-complete problem classified in [5]. The problem of reducing the number of crossings have been studied in many areas. The most prominent areas are VLSI-layouts. Introduction of the VLSI technology revolutionized circuit design and had a strong impact on parallel computing. A lot of research aiming at efficient use of the new technologies has been done and further investigations are in progress. As a crossing of two edges of the communication graph requires unit area in its VLSI-layout, the crossing number together with the number of vertices of the graph immediately provide a lower bound for the area of the VLSI-layout of the communication graph. The crossing numbers has been also studied to improve the readability of hierarchical structures and automated graph drawings. For the understandability of graph drawings, the reducing of crossings is by far the most important.

In the paper, we will use notations and definitions of the crossing numbers of graphs like in [9]. We will often use the Kleitman's result [7] on crossing numbers of the complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

Using Kleitman's result [7], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [9]. Moreover, the exact values for crossing numbers of $G + D_n$ and $G + P_n$ for all graphs G of order at most four are given in [13]. Also, the crossing numbers of the graphs $G + D_n$ are known for few graphs G of order five and six in [3], [8], [11], [12], [15], [16], and [17]. In all of these cases, the graph G is mostly connected and contains also mostly at least one cycle. Further, the exact values for the crossing numbers $G + P_n$ and $G + C_n$ have been also investigated for some graphs G of order five and six in [8], [10], [12], [14], and [18].

The methods presented in the paper are new, and they are based on combinatorial properties of the cyclic permutations. Similar methods were partially used for the first time in the papers [6], and [15]. In [3], [4], [16], and [17], the properties of cyclic permutations were also verified with the help of software in [2].

2. The crossing number of $G + D_n$

Let G be the connected graph of order five isomorphic with the complete bipartite graph $K_{2,3}$. We will consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, $T^1 \cup T^2 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G + D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

In the paper, we will use the same notation and definitions for cyclic permutations for a good drawing D of the graph $G + D_n$ like in [4], and [16]. Let D be a drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D like the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i has been defined by Hernández-Vélez, Medina, and Salazar [6]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We have to emphasize that a rotation is a cyclic permutation. In the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. We will deal with the minimal necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ depending on their rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$. Let us separate all subgraphs T^i, $i = 1, \dots, n$, of the graph $G + D_n$ into three mutually-disjoint subsets depending on how many times the considered T^i crosses the edges of G in D. For $i = 1, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses the edges of G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given subdrawing of G in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

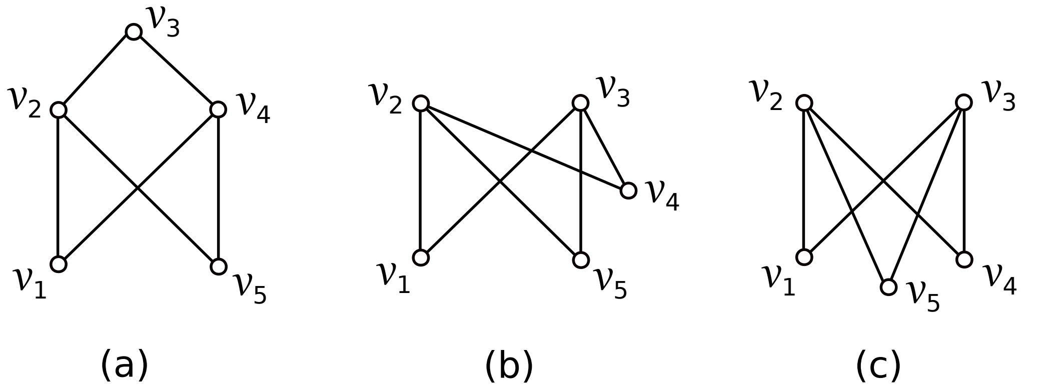

Figure 1: Three drawings of G with a possibility of an existence of a subgraph T^i which do not cross the edges of G

According to the arguments in the proof of the main Theorem 1, if we would like to obtain a drawing of $G + D_n$ with the smallest number of crossings, then the set R_D must be nonempty. Hence, we will deal with only drawings of the graph G with a possibility of an existence of a subgraph T^i, which do not cross the edges of G, i.e., $T^i \in R_D$. The reader can easily verify that there are only three possibilities of drawings of G with the desired property according to the considered good subdrawing of G. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Figure 1(a), (b), and (c). These vertex notations of the graph G will be justified later.

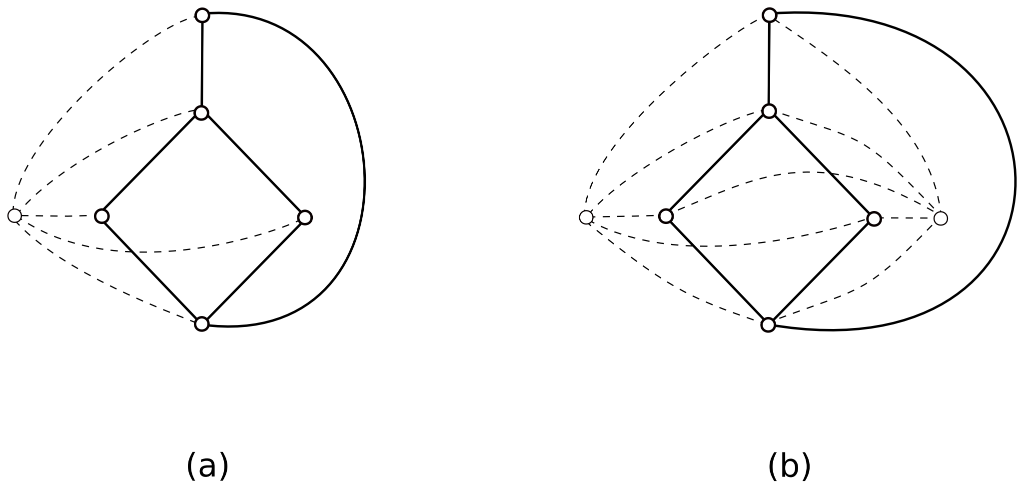

Figure 2: The good drawings of $G + D_1$ and of $G + D_2$

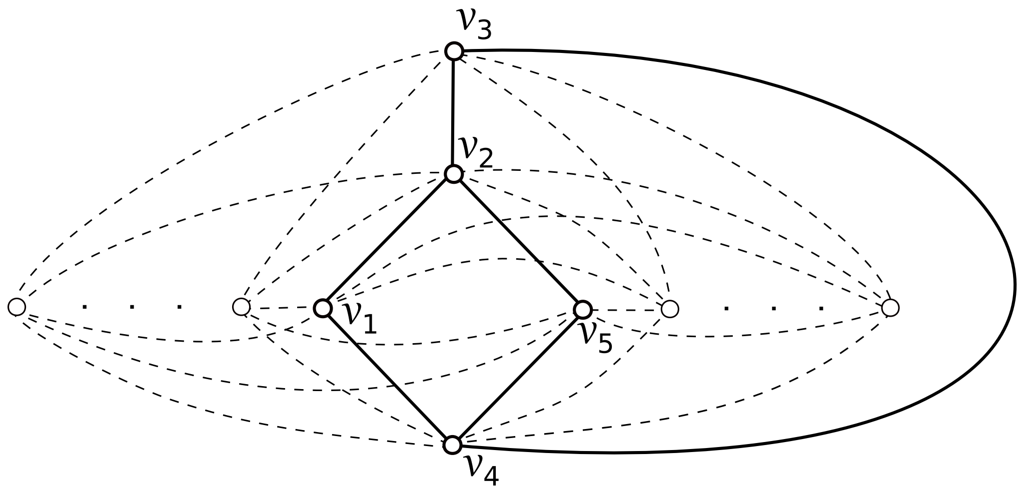

Figure 3: The good drawing of $G + D_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n$ crossings

Two vertices t_i and t_j of $G + D_n$ are *antipodal* in a drawing of $G + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. In the rest of the paper, each considered drawing of the graph $G + D_n$ will be assumed antipodal-free. Now we are able to prove the main result of the paper.

Theorem 1. $\mathrm{cr}(G+D_n) = 4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + n$ *for any* $n \geq 1$.

Proof. In Figure 3 there is the drawing of the graph $G+D_n$ with $4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + n$ crossings. Thus, $\mathrm{cr}(G+D_n) \leq 4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + n$. We prove the reverse inequality by induction on n. The graph $G+D_1$ contains a subdivision of $K_{3,3}$, and therefore $\mathrm{cr}(G+D_1) \geq 1$. So, $\mathrm{cr}(G+D_1) = 1$ by the good drawing of $G+D_1$ in Figure 2(a). Since the graph $G+D_2$ contains a subdivision of $K_{3,4}$, we have $\mathrm{cr}(G+D_2) \geq 2$. Hence, $\mathrm{cr}(G+D_2) = 2$ by the good drawing of $G+D_2$ in Figure 2(b). Suppose now that for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G+D_n) < 4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + n, \tag{2}$$

and let

$$\mathrm{cr}(G+D_m) \geq 4\left\lfloor \frac{m}{2}\right\rfloor \left\lfloor \frac{m-1}{2}\right\rfloor + m \quad \text{for any integer } m < n. \tag{3}$$

Let us first show that the considered drawing D must be antipodal-free. As a contradiction, suppose that, without loss of generality, $\mathrm{cr}_D(T^n, T^{n-1}) = 0$. Using possible subdrawings of G in Figure 1, one can easily verify that the subgraphs T^n and T^{n-1} are not from the set R_D, i.e., $\mathrm{cr}_D(G, T^n \cup T^{n-1}) \geq 2$. The known fact that $\mathrm{cr}(K_{5,3}) = 4$ implies that any T^k, $k = 1, \ldots, n-2$, crosses the edges of the subgraph $T^n \cup T^{n-1}$ at least four times. Therefore, for the number of crossings in the considered drawing D, we have:

$$\begin{aligned}\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D\,(G+D_{n-2}) + \mathrm{cr}_D(T^n \cup T^{n-1}) + \mathrm{cr}_D(G, T^n \cup T^{n-1})\\ +\mathrm{cr}_D(K_{5,n-2}, T^n \cup T^{n-1}) \geq 4\left\lfloor \frac{n-2}{2}\right\rfloor \left\lfloor \frac{n-3}{2}\right\rfloor + n - 2\\ +0+2+4(n-2) = 4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + n.\end{aligned}$$

This contradiction with the assumption (2) confirms that D must be an antipodal-free drawing. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (3) together with the well-known fact $\mathrm{cr}(K_{5,n}) = 4\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$ imply that in D, there is at least one subgraph T^i, which do not cross the edges of G. More precisely:

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{5,n}) \leq \mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) < n,$$

i.e.,

$$s + 2(n-r-s) < n. \tag{4}$$

This forces that $r \geq 1$. In addition, without lost of generality, we can choose the vertex notation of the graph G in such a way as shown in Figure 1(a), (b), and (c).

Now, for a $T^i \in R_D$, the reader can easily verify that the subgraph $F^i = G \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (12345)$, and $\mathrm{cr}_D(T^i, T^j) \geq 4$ for any $T^j \in R_D$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$, for more see [19]. Moreover, one can easily verify over all possible drawings D that $\mathrm{cr}_D(G \cup T^i, T^k) \geq 3$ for any subgraph $T^k \in S_D$. Since there is one crossing among edges of the graph G in all three subdrawings of G in D (in which there is a subgraph $T^i \in R_D$), by fixing the subgraph $G \cup T^i$,

$$\mathrm{cr}(G + D_n) = \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G \cup T^i) + \mathrm{cr}_D(G \cup T^i) \geq$$

$$\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(r-1) + 3s + 3(n-r-s) + 1 = 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor +$$

$$+3n + r - 3 \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 1 - 3 \geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n.$$

Thus, it was shown that there is no good drawing D of the graph $G + D_n$ with less than $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n$ crossings. This completes the proof of Theorem 1. □

3. Corollary

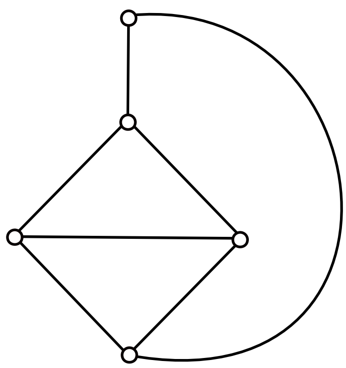

Figure 4: The graph H by adding one edge to the graph G

Let H be the graph obtained from G by adding the edge v_1v_5 in the subdrawing in Figure 1(a). Since we are able to add this edge to the graph G without additional crossings in Figure 3, the drawing of the graph $H + D_n$ with $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + n$ crossings is obtained. Thus, the next result is obvious.

Corollary 1. $\mathrm{cr}(H + D_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n$ *for any* $n \geq 1$.

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