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On the join products of two special graphs on five vertices with the path and the cycle

Michal Staš[1] and Jana Petrillová[2]

Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics Technical University of Košice, Letná 9, 042 00 Košice, Slovak Republic

e-mail: [1] michal.stas@tuke.sk, [2] jana.petrillova@tuke.sk

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Abstract. The investigation on the crossing numbers of graphs is very difficult problem provided that an computing of the crossing number of a given graph in general is NP-complete problem. The problem of reducing the number of crossings in the graph is studied not only in the graph theory, but also by computer scientists. The exact values of the crossing numbers are known only for some graphs or some families of graphs. In the paper, we extend known results concerning crossing numbers for join products of two graphs of order five with the path P_n and the cycle C_n on n vertices. The methods used in the paper are new, and they are based on combinatorial properties of cyclic permutations.

Keywords: graph, drawing, crossing number, join product, cyclic permutation

MSC numbers: 05C10, 05C38

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1. Introduction

The problem of reducing the number of crossings was studied in many areas. The most prominent areas is VLSI-layouts. Introduction of the VLSI technology revolutionized circuit design and had a strong impact on parallel computing. A lot of research aiming at efficient use of the new technologies has been done and further investigations are in progress. As a crossing of two edges of the communication graph requires unit area in its VLSI-layout, the crossing number together with the number of vertices of the graph immediately provide a lower bound for the area of the VLSI-layout of the communication graph. The crossing numbers has been also studied to improve the readability of hierarchical structures and automated graph drawings. The visualized graph should be easy to read and understand. For the understandability of graph drawings, the reducing of crossings is by far the most important.

In the paper, we will deal with determining of the crossing numbers of the join products of two graphs. Let G be a simple graph with vertex set V and edge set E. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a good drawing of G in the plane. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross each other. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$. Let D_n consist on n isolated vertices, P_n and C_n be the path and the cycle on n vertices, respectively. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

It was proved by Garey and Johnson [4] that computing the crossing number of a graph is an NP-complete problem. The exact values of crossing numbers are known only for a few specific families of graphs. Specially for join product it was proved the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle in [8]. Moreover, the exact values for crossing numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most four are given in [12], and the crossing numbers of the graphs $G + D_n$, $G + P_n$, and $G + C_n$ are also known for some graphs G of order five and six, see [9], [11], [13], and [15]. We extend known results concerning crossing numbers for join products of two graphs G, and H of order five with the path P_n and the cycle C_n on n vertices. The disconnected graph G consists of one 4-cycle and of one isolated vertex (see Figure 1(a)), and the connected graph H consists of one 3-cycle and of two non-adjacent leaves (see Figure 1(c)). The methods used for establishing the crossing numbers of $G + D_n$ for disconnected graphs are new, and they are based on combinatorial properties of cyclic permutations. The similar methods were used first time in the papers [2], [3], [15], and [16]. In [1], the properties of cyclic permutations are verified by applying computer programs.

Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

In the paper, some proofs are based on the Kleitman's result [7] on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

2. The crossing number of $G + P_n$ and $G + C_n$

We consider the join product of G with the discrete graph on n vertices D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where any vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, $T^1 \cup T^2 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G + D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

Let D be a good drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [5]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We emphasize that a rotation is a cyclic permutation. For $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, every subgraph $T^i \cup T^j$ of the graph $G + D_n$ is isomorphic with the graph $K_{5,2}$. D. R. Woodall [17] defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \ldots, P_{24}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i

and P_j, which are formed by the exchange of exactly two adjacent elements of the 5-tuple (i. e. an ordered set with 5 elements). Hence, if $d_{COG}(\text{"rot}_D(t_i)\text{"}, \text{"rot}_D(t_j)\text{"})$ denotes the distance between two vertices correspond to the cyclic permutations $\text{rot}_D(t_i)$ and $\text{rot}_D(t_j)$ in the graph COG, then

$$d_{COG}(\text{"rot}_D(t_i)\text{"}, \text{"}\overline{\text{rot}_D(t_j)}\text{"}) = Q(\text{rot}_D(t_i), \text{rot}_D(t_j)) \leq \text{cr}_D(T^i, T^j)$$

for any two different subgraphs T^i and T^j, where $Q(\text{rot}_D(t_i), \text{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\text{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\text{rot}_D(t_j)$. As the complete bipartite graph $K_{5,n}$ is a subgraph of $G+D_n$, we will use the following properties of crossings among edges of its subgraph $K_{5,2}$ with the help of Woodall's results [17] in the subdrawing of $T^i \cup T^j$ induced by any good and antipodal free drawing D of $K_{5,n}$:

1. If $\text{rot}_D(t_i) = \text{rot}_D(t_j)$, then $\text{cr}_D(T^i, T^j) \geq 4$.
2. $\text{cr}_D(T^i, T^j) \geq 4 - Q\left(\text{rot}_D(t_i), \overline{\text{rot}_D(t_j)}\right)$.
3. $\text{cr}_D(T^i \cup T^j, T^k) \geq Q\left(\text{rot}_D(t_i), \overline{\text{rot}_D(t_j)}\right)$ in the subdrawing of $T^i \cup T^j \cup T^k$ induced by D for any $k \neq i, j$.
4. $\text{cr}_D(T^i, T^j) = Q(\text{rot}_D(t_i), \text{rot}_D(t_j)) + 2k$ for some non-negative integer k.

In [15], it was proved that $\text{cr}(G + D_n) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 1$. We will deal with the minimum necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced by the drawing D of the graph $G + P_n$ depending on the rotations $\text{rot}_D(t_i)$ and $\text{rot}_D(t_j)$. We will separate the subgraphs T^i, $i = 1, \ldots, n$, of the graph $G + P_n$ into three subsets depending on how many the considered T^i crosses the edges of G in D. For $i = 1, 2, \ldots, n$, let $R_D = \{T^i : \text{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \text{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses G at least twice in D. We denote by $r = |R_D|$ and $s = |S_D|$. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \ldots, n\}$. Thus, any F^i is exactly represented by $\text{rot}_D(t_i)$.

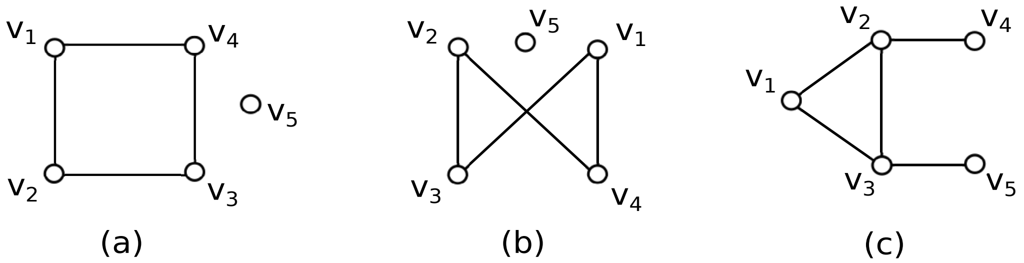

Figure 1: Two possible drawings of G with the vertex notations and drawing of H

Assume a good drawing D of the graph $G + P_n$ in which the edges of G do not cross each other. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Figure 1(a). There are only four different possible configurations of F^i summarized in Table 1 (see [15]). In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. We denote by $\mathcal{M}$ the set of all configurations that exist in the drawing D belonging to the set $\mathcal{M} = \{A_1, A_2, A_3, A_4\}$.

$A_1 : (15432)$	$A_2 : (14325)$
$A_3 : (14352)$	$A_4 : (14532)$

Table 1: Configurations of graph $G \cup T^i$ with vertices denoted of G as in Fig. 1(a)

Let X, Y be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(X, Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X, Y) = \min\{\mathrm{cr}_D(X, Y)\}$ over all good drawings of the graph $G + P_n$. All lower-bounds of number of crossing of configurations from $\mathcal{M}$ are summarized in Table 2 (see [15]).

$-$	A_1	A_2	A_3	A_4
A_1	4	3	2	3
A_2	3	4	3	2
A_3	2	3	4	3
A_4	3	2	3	4

Table 2: The necessary number of crossings between T^i and T^j for the configurations A_k, A_l of subgraphs F^i and F^j, where $k, l \in \{1, 2, 3, 4\}$

Similarly, there is only one drawing of the graph G with one crossing among its edges and with a possibility of an existence of a subgraph T^i which do not cross the edges of G. Assume now a good drawing D of the graph $G + P_n$ in which the edges of G cross once as shown in Figure 1(b). Then, in the drawing D, we obtain the same lower-bounds of number of crossing of two configurations like in the previous case (see [15]).

Now we are able to prove the main results of the paper. We will compute the exact values of crossing numbers of the small graphs $G + P_2$, $G + C_3$, and $H + C_3$ in this paper using the algorithm located on the website `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs, as described in [6]. Unfortunately, the capacity of this system is restricted.

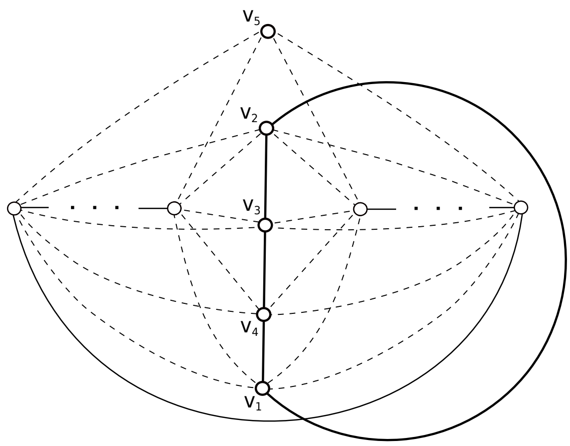

Figure 2: A drawing of $G+P_n$

Theorem 1. $\mathrm{cr}(G+P_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ *for* $n \geq 2$.

Proof. In Figure 2 there is a drawing of $G+P_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings. Thus, $\mathrm{cr}(G+P_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$. We prove the reverse inequality by induction on n. Using algorithm on the website, we can prove that the result is true for $n=2$. Suppose now that, for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G+P_n) < 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1, \tag{2}$$

and let

$$\mathrm{cr}_D(G+P_m) \geq 4\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor + 1 \qquad \text{for any integer } m < n. \tag{3}$$

As the graph $G+D_n$ is a subgraph of the graph $G+P_n$, then $\mathrm{cr}_D(G+P_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$, and therefore, no edge of the path P_n is crossed in D. Let us show that the considered drawing D must be antipodal-free. As a contradiction suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Since the graph $G \cup T^{n-1} \cup T^n$ contains $K_{4,3}$ as a subgraph, and $\mathrm{cr}(K_{4,3}) = 2$, we give

$$2 \leq \mathrm{cr}_D(G \cup T^{n-1} \cup T^n) = \mathrm{cr}_D(G) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(G, T^{n-1} \cup T^n).$$

It implies from the fact $\mathrm{cr}_D(G) \leq 1$, that

$$1 \leq \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(G, T^{n-1} \cup T^n).$$

The known fact $\mathrm{cr}(K_{5,3}) = 4$ implies that any T^k, $k = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least four times. So, for the number of crossings, in D, we have

$$\mathrm{cr}_D(G+P_n)=\mathrm{cr}_D\left(G+P_{n-2}\right)+\mathrm{cr}_D(T^{n-1}\cup T^n)+\mathrm{cr}_D(K_{5,n-2},T^{n-1}\cup T^n)+$$

$$+\,\mathrm{cr}_D(G,T^{n-1}\cup T^n)\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+\left\lfloor\frac{n-2}{2}\right\rfloor+1+4(n-2)+1=$$

$$=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor+1.$$

This contradiction confirms that D is antipodal-free. Moreover, our assumption on D together with $\mathrm{cr}(K_{5,n})=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ implies that

$$\mathrm{cr}_D(G)+\mathrm{cr}_D(G,K_{5,n})\leq\left\lfloor\frac{n}{2}\right\rfloor.$$

Thus,

$$\mathrm{cr}_D(G)+0r+1s+2(n-r-s)\leq\left\lfloor\frac{n}{2}\right\rfloor. \tag{4}$$

Since $\mathrm{cr}_D(G)\leq 1$, we will discuss two following cases:

Case 1: $\mathrm{cr}_D(G)=0$. Then it follows from condition (4) that $s\leq\left\lfloor\frac{n}{2}\right\rfloor$, and $r\geq\left\lceil\frac{n}{2}\right\rceil$. This forces that $r\geq 2$. Now, we will discuss for two different $T^i,T^j\in R_D$ the existence of possible configurations of subgraphs F^i, F^j in the drawing D.

(a) $\{A_1,A_2\}\subseteq\mathcal{M}_D$ or $\{A_1,A_4\}\subseteq\mathcal{M}_D$ or $\{A_2,A_3\}\subseteq\mathcal{M}_D$ or $\{A_3,A_4\}\subseteq\mathcal{M}_D$. Without lost of generality, let us fix two T^{n-1}, $T^n\in R_D$ such that F^{n-1}, F^n have different configurations from $\{A_1,A_2\}\subseteq\mathcal{M}_D$. Let us note that the configurations A_1 and A_2 are represented by the cyclic permutations (15432) and (14325), respectively. Since (14325) can be obtained from (15432) by one interchange of adjacent elements 1 and 5, then $\mathrm{cr}_D(T^{n-1},T^n)\geq 4-1=3$. Using the properties of the cyclic-ordered graph COG mentioned above, it is easy to verify, that $\mathrm{cr}_D(T^{n-1}\cup T^n,T^i)\geq 3$ for any T^i, $i\neq n-1,n$. Moreover, $\mathrm{cr}_D(T^{n-1}\cup T^n,T^i)\geq 5$ for any $T^i\in R_D$, $i\neq n-1,n$ by summing the values in all columns in the first two rows of Table 2. Thus, if we fix the graph $G\cup T^{n-1}\cup T^n$, then the fact $-s\geq-\left\lfloor\frac{n}{2}\right\rfloor$ implies that

$$\mathrm{cr}(G+P_n)\geq\mathrm{cr}_D(K_{5,n-2})+\mathrm{cr}_D(K_{5,n-2},G\cup T^{n-1}\cup T^n)+\mathrm{cr}_D(G\cup T^{n-1}\cup T^n)\geq$$

$$\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+5(r-2)+4s+5(n-r-s)+3=4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+$$

$$+5n-s-7\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+5n-\left\lfloor\frac{n}{2}\right\rfloor-7>4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

Due to symmetry, we can apply the same idea for other mentioned above couples of configurations.

(b) $\mathcal{M}_D = \{A_1, A_3\}$ or $\mathcal{M}_D = \{A_2, A_4\}$.
Assume that $\mathcal{M}_D = \{A_1, A_3\}$. If there are two subgraphs $T^i,\ T^j \in R_D$ with $\mathrm{cr}_D(T^i, T^j) \neq 2$ such that $F^i,\ F^j$ have different configurations from $\{A_1, A_3\}$, then $\mathrm{cr}_D(T^i, T^j) \geq 4$ by Woodall's result (see property 4).

In the case, if there are no two subgraphs $T^i,\ T^j \in R_D$ with $\mathrm{cr}_D(T^i, T^j) = 2$ such that $F^i,\ F^j$ have different configurations from $\{A_1, A_3\}$, then by fixing of the graph $G \cup T^l$, for any $T^l \in R_D$, we have

$$\mathrm{cr}(G + P_n) \geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G \cup T^l) + \mathrm{cr}_D(G \cup T^l) \geq$$

$$\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(r-1) + 2(n-r) + 0 = 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2n +$$

$$+2r - 4 \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2n + 2\left\lceil\frac{n}{2}\right\rceil - 4 > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

In addition, without lost of generality, let us fix two $T^{n-1},\ T^n \in R_D$ such that $F^{n-1},\ F^n$ have different configurations from $\{A_1, A_3\}$ with $\mathrm{cr}_D(T^{n-1}, T^n) = 2$. As no edge of the path P_n is crossed in D, no vertex t_i is placed inside the 4-cycle of the graph G. Moreover, if a vertex t_i is placed in some triangular region of the subdrawing $D(F^n)$ with two vertices of G on its boundary, then $\mathrm{cr}_D(G \cup T^n, T^i) \geq 3$. Hence, by fixing of $G \cup T^n$ we obtain a contradiction

$$\mathrm{cr}(G + P_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3(n-1) + 0 > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

By the same arguments, no vertex t_i is placed inside the triangular region of the subdrawing $D(F^{n-1})$ with two vertices of G on its boundary. Since we assume that $\mathrm{cr}_D(T^{n-1}, T^n) = 2$, then any subgraph T^i, $i \neq n-1, n$ crosses the edges of $T^{n-1} \cup T^n$ at least four times in D, i.e. $\mathrm{cr}_D(T^{n-1} \cup T^n, T^i) \geq 4$. Further, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^j) \geq 4+2 = 6$ for any $T^j \in R_D$, $j \neq n-1, n$. Thus, by fixing of the graph $G \cup T^{n-1} \cup T^n$ we have

$$\mathrm{cr}(G+P_n) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 6(r-2) + 5(n-r) + 2 = 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor +$$

$$+5n + r - 10 \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 5n + \left\lceil\frac{n}{2}\right\rceil - 10 > 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

(c) $\mathcal{M}_D = \{A_i\}$ for some $i \in \{1, 2, 3, 4\}$.
Without lost of generality, we can assume the configuration A_1 of the subgraph $F^l = G \cup T^l$ for some $T^l \in R_D$. By fixing of the graph $G \cup T^l$ we obtain the same inequalities like at the beginning in **C**ase 1b).

Case 2: $\mathrm{cr}_D(G) = 1$. Then it follows from condition (4) that $s < \lfloor \frac{n}{2} \rfloor$, and $r > \lceil \frac{n}{2} \rceil$. This forces that $r \geq 2$. If we apply the same arguments like in **C**ase 1, then

$$\mathrm{cr}(G + P_n) > 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

So, we obtain a contradiction with the assumption that there are less than $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings in the considered drawing D in all mentioned cases. □

The following theorem gives us the exact value of the crossing number of the graph $G + C_n$ for $n \geq 3$.

Theorem 2. $\mathrm{cr}(G + C_n) = 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 3$.

Proof. The edge created the cycle C_n can be added into the drawing of $G + P_n$ in Figure 2 in such a way that the drawing of $G + C_n$ with $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings is obtained. So, $\mathrm{cr}(G+C_n) \leq 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$. Using the algorithm on the website `http://crossings.uos.de/`, we can prove that $\mathrm{cr}(G + C_3) = 7$. Let us assume that there is a good drawing D of the graph $G + C_n$ with less than $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings for $n \geq 4$. Thus, at most one edge of C_n can be crossed and moreover, the edges of C_n do not cross each other (see [10]). Assume the subgraph $D_4 + C_n$ of $G + C_n$, where D_4 consists of the vertices v_1, v_2, v_3, and v_4 (see Figure 1(a)). For $x = v_1, v_2, v_3, v_4$, let T^x denote the subgraph of $G + C_n$ induced by n edges incident with the vertex x.

If none edge of C_n is crossed, then the whole graph G is placed in the same region in the view of the subdrawing of C_n and so, the edges of $T^{v_1} \cup T^{v_2} \cup T^{v_3} \cup T^{v_4}$ cross each other at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times (see [10]). As $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor > 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 4$, we have a contradiction with the number of crossing in D.

If the cycle C_n is crossed once, in the considered drawing D, then there are at least $\binom{4}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings, and we obtain the same contradiction like above. Thus, in all considered drawings of $G+C_n$ there are more than $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings for $n \geq 4$. This completes the proof. □

3. The crossing number of $H + P_n$ and $H + C_n$

Let us consider the graph H which consists of one 3-cycle and of two non-adjacent leaves (see Figure 1(c)). In [2], it was proved that $\mathrm{cr}(H + D_n) = 4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ for $n \geq 1$ also with the help of the combinatorial properties of cyclic permutations. As we can obtained the drawing of the graph $H+P_n$ with $4 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings from the drawing in Figure 3, the next result is obvious.

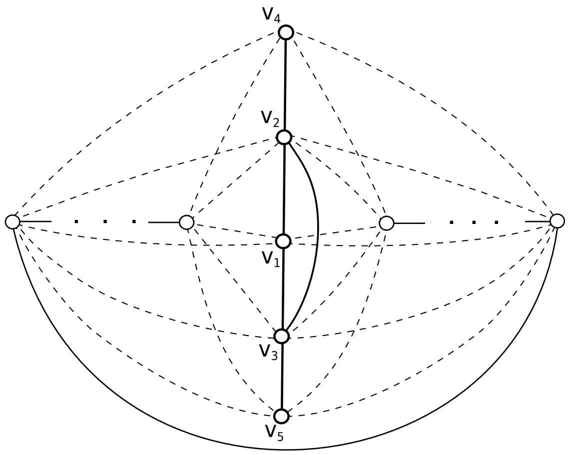

Figure 3: A drawing of $H + P_n$

Theorem 3. $\mathrm{cr}(H + P_n) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 2$.

Into the drawing in Figure 3, it is possible to add the edge which forms the cycle C_n on the path P_n in such a way that C_n is crossed by H just twice. Hence, the crossing number of the graph $H + C_n$ is at most $4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2$. By the same arguments like in Theorem 2, we can prove the reverse inequality for $n \geq 4$. The crossing number of the small graph $H + C_3$ can be computed also using the algorithm located on the website `http://crossings.uos.de/`. These results are collected in the next theorem.

Theorem 4. $\mathrm{cr}(H + C_n) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2$ *for* $n \geq 3$.

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