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Approximate Formulas for Mathematical Expectations of Functionals of Random Processes Defined by Ito-Levy Multiple Integral Expansion

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Abstract. An approximate formula is constructed for calculating mathematical expectations of nonlinear functionals of random processes defined by chaotic expansions in terms of multiple Ito-Levy integrals. We consider the possibility of applying the formula to the calculation of mathematical expectation of functionals corresponding to the solution of one kind of Skorokhod equations on a Wiener process. Test examples of computations using the constructed formula for particular cases of Levy process are presented. The elaborated method gives a new useful tool for numerical integration with a required accuracy of stochastic differential equations which are reduced to the Skorokhod class of equations.

Keywords: functionals of Levy processes, Ito-Levy multiple integral expansion, mathematical expectations of process functionals, approximate formulae.
MSC numbers: 65C30, 60H35

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1. Introduction

One of the main problems in the analysis of stochastic models of dynamic systems is the determination of probability and statistical characteristics of underlying random processes, particularly, the characteristics of solutions of the appropriate stochastic differential equations. The functional dependence of the solutions of such equations on the random processes perturbing the system makes it necessary to use the methods for analysing functionals defined on random process trajectories. Practical application of these methods, in turn, requires the development of adequate approximate approaches. For the numerical solution of stochastic differential equations, they most frequently use the well-developed approach based on finite discretisation of the temporal interval and statistical modelling of the solution at discrete moments of time [1, 2]. However, as a rule, the calculation of functionals defined on the trajectories of solutions of stochastic differential equations requires the use of the random process values in the entire domain of definition. In this case, the efficiency of using the solution values found by means of statistical simulation at the fixed moments of time determined by the discrete scheme is low efficient. Hence, the problem of developing an alternative approach based on approximating the process functionals as functionals defined on trajectories and applying the approximate methods of functional analysis is rather urgent. In particular, to calculate the mathematical expectations of random processes determined by the solutions of stochastic differential equations (in particular, when the explicit form of the solution is known) one can efficiently use functional quadrature formulae similar to those derived to calculate functional integrals (see [3, 4, 5]).

In the present paper, we derive the formulae of such type for calculating the mathematical expectations of nonlinear functionals of random process, defined by chaotic expansions in multiple stochastic Ito integrals. We study the case, rather general and important for applications, namely, the functionals of the processes having a continuous component and a discontinuous one (the functionals of Levy processes). The consideration of this class of processes was motivated also by the existence of the formula for a product of multiple stochastic integrals of Levy process [6, 7], which was used to derive the approximate formula of the present work. The authors of Refs. [8, 9] considered the application of chaotic expansions to the calculation of mathematical expectations of some kinds of functionals of Wiener and Poisson processes. There is a number of publications on chaotic expansions of random process functionals in multiple stochastic integrals, including their use in stochastic analysis, in different fields of physics, and in financial mathematics. The definitions and information necessary for further considerations can be found in [6, 7, 10, 11, 12].

The paper is organised as follows. In Section 2, we present the derivation of the approximate formula. In Section 3, we estimate the accuracy of the derived formula. In Section 4, we present the application of the formula to the stochastic Skorokhod equation for Wiener process and illustrating numerical examples.

2. Derivation of approximate formula

Consider the random process X_t, $t \in [0,1]$, that allows the expansion in multiple integrals [6, 7]

$$X_t = \sum_{n=0}^{N} I_n(x_{t,n}), \tag{1}$$

where $I_n(x_{t,n}) = \int\limits_U (n) \int\limits_U x_{t,n}(u_1,...,u_n)dM(u_1)...dM(u_n)$ is a random measure multiple integral on $U = [0,1] \times R$, $M(U) = \int\limits_0^1 1_U(\tau,0)dW_\tau + \int\limits_0^1 \int\limits_{R_0} z 1_U(\tau,z)d\tilde{N}(\tau,z)$, $U \in \mathcal{B}([0,1] \times R_0)$, $R_0 = R \backslash \{0\}$; $\mathcal{B}([0,1] \times R_0)$ is the Borel σ–algebra, $\tilde{N}$ is the compensated random measure, determined by the equality $\tilde{N}([0,\tau] \times A) = N([0,\tau] \times A) - \tau v(A)$ is the Poisson random measure, $v(A)$ is the Levy measure satisfying the condition $\int\limits_R (1 \wedge z^2)\, dv(z) < \infty$. Note that M defines the centred integrable Levy process $Y_t = M([0,t] \times R_0) = W_t + \int\limits_0^t \int\limits_{R_0} z d\tilde{N}(\tau,z)$.

The following formulae are valid [6, 7]

$$E[X_{t_1} X_{t_2}] = \sum_{n=0}^{N} n! \int\limits_U (n) \int\limits_U x_{t_1,n}(u_1,...,u_n) x_{t_2,n}(u_1,...,u_n) d\lambda(u_1)...d\lambda(u_n), \tag{2}$$

where $E[\,]$ denotes the mathematical expectation,

$$I_n(x_{t_1,n}) I_m(x_{t_2,m}) = \sum_{k=0}^{n\wedge m} \sum_{r=0}^{(n\wedge m)-k} k! r! C_n^k C_m^k C_{m-k}^r C_{n-k}^r I_{n+m-2k-r}\left(x_{t_1,n} \hat{\otimes}_k^r x_{t_2,m}\right), \tag{3}$$

$x_{t_1,n} \hat{\otimes}_k^r x_{t_2,m}$ denotes the symmetrisation of a function

$$x_{t_1,n} \hat{\otimes}_k^r x_{t_2,m}(u_1,...,u_{n-k},v_1,...,v_{m-k-r}) = \prod_{l=n-k-r+1}^{n-k} p_2(u_l) \times$$
$$\int\limits_{U^k} x_{t_1,n}\Big(u_1,...,u_{n-k-r}, \underline{u_{n-k-r+1},...,u_{n-k}}, u_{n-k-r+1},...,u_{n-k}, u_{n-k+1},...,u_n\Big) \times$$
$$x_{t_2,m}\Big(v_1,...,v_{m-k-r}, \underline{u_{n-k-r+1},...,u_{n-k}}, u_{n-k+1},...,u_n\Big) d\lambda(u_{n-k+1})...d\lambda(u_n),$$

where $u_l = (\tau_l, z_l)$, $p_2(u_l) = z_l$; the underlined symbols denote the variables to which the operator $p_2(u_l) = z_l$ was applied; $d\lambda(\tau,z) = d\tau$ on $\mathcal{B}(R \times \{0\})$, $d\lambda(\tau,z) = z^2 v(dz) d\tau$ on $\mathcal{B}(R^2 \backslash (\tau,0))$.

From Eqs. (2), (3) it directly follows that

$$E\left[X_{t_1}X_{t_2}X_{t_3}\right]=\sum_{n,m=0}^{k}\sum_{k=0}^{n\wedge m}\sum_{r=0}^{(n\wedge m)-k}k!r!C_n^kC_m^kC_{m-k}^rC_{n-k}^r(m+n-2k-r)!\times$$
$$\int\limits_{U^{n+m-2k-r}}^{N}x_{t_1,n}\hat{\otimes}_k^r x_{t_2,m}\left(u_1,...,u_{n-k},v_1,...,v_{m-k-r}\right)\times$$
$$x_{t_3,n+m-2k-r}\left(u_1,...,u_{n-k},v_1,...,v_{m-k-r}\right)d\lambda(u_1)...d\lambda(u_{n-k})d\lambda(v_1,...,v_{m-k-r}). \quad (4)$$

Let us write Eq. (4) in the explicit form for the case $X_{t_j}=\sum_{n=0}^{N}I_n\left(x_{t_j,n}\right)$, $N=2$; $j=1,2,3$:

$$E\left[X_{t_1}X_{t_2}X_{t_3}\right]\approx x_{t_1,0}x_{t_2,0}x_{t_3,0}+$$
$$x_{t_1,0}\sum_{k=1}^{2}k!\int\limits_{U^k}x_{t_2,k}(u_1,...,u_k)x_{t_3,k}(u_1,...,u_k)d\lambda(u_1)...d\lambda(u_k)+$$
$$x_{t_2,0}\sum_{k=1}^{2}k!\int\limits_{U^k}x_{t_1,k}(u_1,...,u_k)x_{t_3,k}(u_1,...,u_k)d\lambda(u_1)...d\lambda(u_k)+$$
$$x_{t_3,0}\sum_{k=1}^{2}k!\int\limits_{U^k}x_{t_1,k}(u_1,...,u_k)x_{t_2,k}(u_1,...,u_k)d\lambda(u_1)...d\lambda(u_k)+$$

$$2\int\limits_{U^2}x_{t_1,1}(u_1)x_{t_2,2}(u_1,u_2)x_{t_3,1}(u_2)d\lambda(u_1)d\lambda(u_2)+$$
$$2\int\limits_{U^2}x_{t_1,1}(u_1)x_{t_2,1}(u_2)x_{t_3,2}(u_1,u_2)d\lambda(u_1)d\lambda(u_2)+$$
$$2\int\limits_{U^2}x_{t_1,2}(u_1,u_2)x_{t_2,1}(u_2)x_{t_3,1}(u_2)d\lambda(u_1)d\lambda(u_2)+$$

$$\int\limits_{U} p_2(u)x_{t_1,1}(u)x_{t_2,1}(u)x_{t_3,1}(u)d\lambda(u)+$$

$$2\cdot 2!\int\limits_{U^2} p_2(u_1)x_{t_1,1}(u_1)x_{t_2,2}(u_1,v_1)x_{t_3,2}(u_1,v_1)d\lambda(u_1)d\lambda(v_1)+$$

$$2\cdot 2!\int\limits_{U^2} p_2(u_2)x_{t_1,2}(u_1,u_2)x_{t_2,1}(u_2)x_{t_3,2}(u_1,u_2)d\lambda(u_1)d\lambda(u_2)+$$

$$2\cdot 2!\int\limits_{U^2} p_2(u_1)x_{t_1,2}(u_1,u_2)x_{t_2,2}(u_2,u_1)x_{t_3,1}(u_1)d\lambda(u_1)d\lambda(u_2)+$$

$$2\cdot 2\cdot 2!\int\limits_{U^3} x_{t_1,2}(u_1,u_2)x_{t_2,2}(u_2,v_1)x_{t_3,2}(u_1,v_1)d\lambda(u_1)d\lambda(u_2)d\lambda(v_1)+$$

$$(2!)^2\int\limits_{U^2} p_2(u_1)p_2(u_2)x_{t_1,2}(u_1,u_2)x_{t_2,2}(u_1,u_2)x_{t_3,2}(u_1,u_2)d\lambda(u_1)d\lambda(u_2).$$

Let us introduce the notations

$$x_1(t)=x_{t,n}\left(u_1,...,u_{n-k-r},\underline{u_{n-k-r+1},...,u_{n-k}},u_{n-k-r+1},...,u_{n-k},u_{n-k+1},...,u_n\right),$$

$$x_2(t)=x_{t,m}\left(v_1,...,v_{m-k-r},\underline{u_{n-k-r+1},...,u_{n-k}},u_{n-k+1},...,u_n\right),$$

$$x_3(t)=x_{t,n+m-2k-r}\left(u_1,...,u_{n-k},v_1,...,v_{m-k-r}\right).$$

The construction of the approximate formula considered in the theorem below is based on the requirement of its exactness for cubic functional polynomials of the process trajectories (see. [3, 4]), i.e., the functionals having the form

$$P_3\left(X_{(\cdot)}\right)=a_0+\sum_{k=1}^{3}\int\limits_{[0,T]}(k)\int\limits_{[0,T]}a_k(t_1,...,t_k)\prod_{l=1}^{k}X_{t_l}dt_1...dt_k,$$

where a_0 is a constant, $a_k(t_1,...,t_k)$, $k=1,2,3$ are given deterministic functions. We assume that the considered functionals of the process trajectories can be approximated by functional polynomials sufficiently well (see Eq. (6) in Section 2).

Theorem 1. *Let the functions $x_t(u_1,...,u_n)$ in the expansion (1) be differentiable with respect to t, $f_0(u_1,...,u_n)\neq 0$, $f_s(u_1,...,u_n)\neq 0$. Then the following approximate* *formula, exact for symmetric functional cubic polynomials* $X_{(\cdot)}$ *is valid,*

$$E\left[F\left(X_{(\cdot)}\right)\right] \approx J_N\left(F\left(X_{(\cdot)}\right)\right) \equiv \sum_{n,m=0}^{N}\sum_{k=0}^{n\wedge m}\sum_{r=0}^{(n\wedge m)-k} C_{n,m,k,r}\times$$

$$\sum_{k=1}^{2} A_k \int\limits_{U^{n+m-k-r}} \prod_{l=n-k-r+1}^{n-k} p_2(u_l) J_{n,m,k,r}\left(F(\cdot), c_k, u, v, y, z\right) d\lambda(u_1)...d\lambda(u_{n-k-r})\times$$

$$d\lambda(v_1)...d\lambda(v_{m-k-r})d\lambda(y_1)...d\lambda(y_k)d\lambda(z_1)...d\lambda(z_r)+$$

$$\sum_{n=0}^{N} b_n \int\limits_{U^n} \Delta F\left(b_n^{-\frac{1}{2}} x_{\cdot,n}(u_1,...,u_n)\right) d\lambda(u_1)...d\lambda(u_n) + F(0)(1-B), \quad (5)$$

where

$$J_{n,m,k,r}(F) =$$

$$-\int_0^1\int_0^1 \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right)\frac{\partial}{\partial \tau}\left(\frac{x_3(\tau)}{x_2(\tau)}\right)\right)^{1/3} 1_{[0,\cdot]}(s)1_{[\cdot,1]}(\tau)\right) ds d\tau -$$

$$\frac{x_1(0)}{x_2(0)}\int_0^1 \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_3(s)}{x_2(s)}\right)\right)^{1/3} 1_{[\cdot,1]}(s)\right) ds +$$

$$\frac{x_3(1)}{x_2(1)}\int_0^1 \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right)\right)^{1/3} 1_{[0,\cdot]}(s)\right) ds + \frac{x_1(0)}{x_2(0)}\frac{x_3(1)}{x_2(1)}\Lambda F\left(c_k x_2(\cdot)\right),$$

if there are no coincident elements among $x_1(t)$, $x_2(t)$, $x_3(t)$*;*

$$J_{n,m,k,r}(F) = \int_0^1 \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_3(s)}{x_2(s)}\right)\right)^{1/3} 1_{[0,\cdot]}(s)\right) ds + \frac{x_1(0)}{x_2(0)}\Lambda F\left(c_k x_2(\cdot)\right),$$

if $x_1(t) \equiv x_2(t)$*,*

$$J_{n,m,k,r}(F) = \int_0^1 \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_3(s)}{x_2(s)}\right)\right)^{1/3} 1_{[\cdot,1]}(s)\right) ds + \frac{x_3(1)}{x_2(1)}\Lambda F\left(c_k x_2(\cdot)\right),$$

if $x_3(t) \equiv x_2(t)$*;* $J_{n,n,n,n}(F) = \Lambda F\left(c_k x_1(\cdot)\right)$*;* $C_{n,m,k,r} = k!r!C_n^k C_m^k C_{m-k}^r C_{n-k}^r (m+n-2k-r)!$*,* C_n^k *is the combination of* n *elements taken* k *at a time;* $F(\cdot)$ *denotes the expression determined by the operator* $J_{n,m,k,r}$*;* A_k*,* c_k*,* $k = 1,2,3$ *are the constants that satisfy the system of equations* $A_1c_1 + A_2c_2 = 0$*,* $A_1c_1^3 + A_2c_2^2 = 1$*;* $\Delta F(f) = \frac{1}{2}\left(F(f) + F(-f)\right)$*,* $B = \sum\limits_{n=0}^{N} b_n$*,* $b_n > 0$*.*

The theorem is proved by direct calculation of the left-hand side of Eq. (5) and the right-hand one for the functionals $F\left(X_{(\cdot)}\right) = const$, $F\left(X_{(\cdot)}\right) = X_t$, $F\left(X_{(\cdot)}\right) =$

$X_{t_1}X_{t_2}$, $F\left(X_{(\cdot)}\right) = X_{t_1}X_{t_2}X_{t_3}$. Note that in this case all integrals in the right-hand side are calculated exactly. Since the triple sum resulting from the expansion of the monomial $X_{t_1}X_{t_2}X_{t_3}$ factors in the series (1) can be presented as a sum of blocks, consisting of the sums $\sum_{\{n,m,p\}}$, in which the set of indices $\{n, m, p\}$ is fixed and the summation is executed over all possible permutations within this set, the result of calculation in the right-and side of Eq.(5) is independent of the order of t_1, t_2, t_3. Therefore, in the calculation of individual terms in Eq. (5) it is sufficient to assume $t_1 \leq t_2 \leq t_3$.

Note. If the initial process expands in an infinite series, the approximately exact formulae are considered, in which a finite segment of the chaotic expansion is used.

3. Accuracy estimation for the derived approximate formula

Here and below, we use the class of functionals that allow representation in the form

$$F\left(X_{(\cdot)}\right) = F(0) + \sum_{k=1}^{3} \frac{1}{k!} \int\limits_{[0,T]} (k) \int\limits_{[0,T]} A_k(t_1, ..., t_k) \prod_{l=1}^{k} X_{t_l} dt_1 ... dt_k +$$
$$\frac{1}{3!} \int\limits_0^1 (1-\tau)^3 \int\limits_{[0,T]} (4) \int\limits_{[0,T]} G\left(\tau X_{(\cdot)}, t_1, ..., t_4\right) \prod_{l=1}^{4} X_{t_l} dt_1 ... dt_4. \qquad T < 1, \quad (6)$$

where $A_k(t_1, ..., t_k)$ are symmetrised deterministic functions.

Such representation is possible, e.g., for some functionals that have functional derivatives up to the fourth order inclusively, in correspondence with the functional Taylor theorem

$$F\left(X_{(\cdot)}\right) = F(0) + \sum_{k=1}^{3} \frac{1}{k!} \int\limits_{[0,T]} (k) \int\limits_{[0,T]} F^{(k)}\left(X_{(\cdot)}, t_1, ..., t_k\right) \prod_{l=1}^{k} X_{t_l} dt_1 ... dt_k +$$
$$\frac{1}{3!} \int\limits_0^1 (1-\tau)^3 \int\limits_{[0,T]} (4) \int\limits_{[0,T]} F^{(4)}\left(\tau X_{(\cdot)}, t_1, ..., t_4\right) \prod_{l=1}^{4} X_{t_l} dt_1 ... dt_4,$$

$F^{(k)}\left(X_{(\cdot)}, t_1, ..., t_k\right)$ is a functional derivative of F. In particular, the functionals having the form $F\left(X_{(\cdot)}\right) = g\left(\int\limits_0^T \alpha(t) X_t dt\right)$ belong to this class, where $g(u)$, $u \in R$ is a deterministic function having the derivatives of the fourth order.

Let us introduce the notation:

$$c(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{k_1=0}^{n_1 \wedge n_2} \sum_{r_1=0}^{(n_1 \wedge n_2)-k_1} \frac{\left(a(n_1, n_2, k_1, r_1)\right)^2 (n_1 + n_2 - 2k_1 - r_1)!}{2^{2(n_1+n_2)}(n_1! n_2!)^2},$$

$$a(n, m, k, r) = k! r! C_n^k C_m^k C_{n-k}^r C_{m-k}^r.$$

Theorem 2. *Let the functional $F\left(X_{(\cdot)}\right)$ allow the representation (6), where*

$$\sup_{\tau\in[0,1],t_1,...,t_4\in[0,T]} \left|G\left(\tau X_{(\cdot)},t_1,...,t_4\right)\right| \leq A \equiv const, \qquad \sup_{t\in[0,T]} |x_{t,n}| \leq 1/(n!2^n).$$

Then the estimate

$$|R| = \left|E\left[F\left(X_{(\cdot)}\right)\right] - J_N\left(F\left(X_{(\cdot)}\right)\right)\right| < T^4\frac{1}{4!}A\left(c(N) + \sum_{n=0}^{N} b_n^{-1}(n!2^n)^{-4}\right) \qquad (7)$$

is valid.

Proof. Using Eq. (6) and the fact that the formula is exact for symmetric functional cubic polynomials, let us estimate the error $R = E\left[F\left(X_{(\cdot)}\right)\right] - J_N\left(F\left(X_{(\cdot)}\right)\right)$:

$$|R| \leq \frac{1}{3!}\int_0^1 (1-\tau)^3 \int_{[0,T]} (4) \int_{[0,T]} \left[\left|E\left[G\left(\tau X_{(\cdot)},t_1,...,t_4\right)\prod_{l=1}^{4} X_{t_l}\right]\right| + \right.$$
$$\left.\left|J_N\left(G\left(\tau X_{(\cdot)},t_1,...,t_4\right)\prod_{l=1}^{4} X_{t_l}\right)\right|\right] dt_1...dt_k d\tau \leq$$

(we use the inequality $|ab| \leq \frac{1}{2}(a^2+b^2)$, which is valid for any real numbers)

$$\leq \frac{1}{4!}A\left[\frac{1}{2}\int_{[0,T]} (4) \int_{[0,T]} E\left[(X_{t_1}X_{t_2})^2 + (X_{t_3}X_{t_4})^2\right] dt_1...dt_k + \right.$$
$$\left.\int_{[0,T]} (4) \int_{[0,T]} \left|J_N\left(\prod_{l=1}^{4} X_{t_l}\right)\right| dt_1...dt_k\right] \equiv \frac{1}{4!}A(R_1+R_2).$$

Then we have

$$I_{n_1}\left(x_{t_1,n_1}\right) I_{n_2}\left(x_{t_2,n_2}\right) =$$
$$\sum_{k_1=0}^{n_1\wedge n_2} \sum_{r_1=0}^{(n_1\wedge n_2)-k_1} k_1!r_1!C_{n_1}^{k_1}C_{n_2}^{k_1}C_{n_1-k_1}^{r_1}C_{n_2-k_1}^{r_1} I_{n_1+n_2-2k_1-r_1}\left(x_{t_1,n_1}\hat{\otimes}_{k_1}^{r_1} x_{t_2,n_2}\right) \equiv$$
$$\sum_{k_1=0}^{n_1\wedge n_2} \sum_{r_1=0}^{(n_1\wedge n_2)-k_1} a(n_1,n_2,k_1,r_1) I_{n_1+n_2-2k_1-r_1}\left(x_{t_1,n_1}\hat{\otimes}_{k_1}^{r_1} x_{t_2,n_2}\right),$$

$$E\left[(X_{t_1}X_{t_2})^2\right] = \sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{k_1=0}^{n_1\wedge n_2}\sum_{r_1=0}^{(n_1\wedge n_2)-k_1} \left(a(n_1,n_2,k_1,r_1)\right)^2 (n_1+n_2-2k_1-r_1)!\times$$
$$\int_{U^{n_1+n_2-2k_1-r_1}} \left(x_{t_1,n_1}\hat{\otimes}_{k_1}^{r_1} x_{t_2,n_2}\right)^2 d\lambda(u_1)...d\lambda(u_{n_1+n_2-2k_1-r_1})$$

and the analogous expression for $E\left[(X_{t_3}X_{t_4})^2\right]$.

Due to the assumption of the theorem we have $\sup\limits_{t\in[0,T]}|x_{t,n}| \leq 1/(n!2^n)$, we have

$$E\left[(X_{t_1}X_{t_2})^2\right] \leq \sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\sum_{k_1=0}^{n_1\wedge n_2}\sum_{r_1=0}^{(n_1\wedge n_2)-k_1}\frac{(a(n_1,n_2,k_1,r_1))^2\,(n_1+n_2-2k_1-r_1)!}{2^{2(n_1+n_2)}(n_1!n_2!)^2},$$

and then $R_1 \leq T^4c(N)$; $J_{n,m,r}\left(\prod\limits_{l=1}^{4}X_{t_l}\right) = 0$, since the functional is even and due to the action of the operator Λ,

$$\left|J\left(\prod_{l=1}^{4}X_{t_l}\right)\right| = \left|\sum_{n=0}^{N}b_n\int\limits_{U^n}b_n^{-2}\prod_{l=1}^{4}x_{t_l,n}(u_1,...,u_n)d\lambda(u_1)...d\lambda(u_n)\right| \leq \sum_{n=0}^{N}b_n^{-1}(n!2^n)^{-4}.$$

$$R_2 = \int\limits_{[0,T]}(4)\underset{\cdots}{}\int\limits_{[0,T]}\left|J\left(\prod_{l=1}^{4}X_{t_l}\right)\right|dt_1...dt_k \leq T^4\frac{1}{4!}A\left(c+\sum_{n=0}^{N}b_n^{-1}(n!2^n)^{-4}\right).$$

From the obtained estimates for R_1, R_2 the statement of the theorem follows.

Note that the series in the right-hand side of Eq. (7) converges at $N\to\infty$.

4. Application to the stochastic Skorokhod equation

A natural class of stochastic differential equations to which the approximate formulae of the considered type could be applied is the Skorokhod class of equations, for which the expansion in multiple Ito-Wiener integrals is used directly in the formulation of the equations themselves. However, when the solutions of such equations are presented as a chaotic series, the division-by-zero singularities arise in the multiple integrals of the coefficient functions. Therefore, direct application of the formulae derived above is impossible. Below by the example in which the chaotic expansion in Ito-Wiener multiple integrals (a particular case of Ito-Levy integrals) is used, we demonstrate a modification of the approximate formula, in which this singularity is eliminated. Consider the calculation of the mathematical expectation for the functionals of the solution of linear equation with the Skorokhod integral on the Wiener process with the initial condition in the form of stochastic expansion with the fixed number of terms. Generally, the solution of the Skorokhod equation

$$X_t = G_t + \int_0^t \alpha(s)X_s\delta M_s,$$

where $\alpha(s)$ is a deterministic function, $G_t = \sum_{n=0}^{N} I_n\left(g_n(\cdot,t)\right)$, M_t is the martingale, can be presented in the form (see [13]) $X_t = \sum_{n=0}^{\infty} I_n\left(f_n(\cdot,t)\right)$, where

$f_n(\cdot,*,t) = g_n(\cdot,*,t) + \left(f_{n-1}(\cdot,*)\alpha(*)1_{[0,t]}(*)\right)^{sym}$ for $1 \le n \le N$, and $f_n(\cdot,*,t) = \left(f_{n-1}(\cdot,*)\alpha(*)1_{[0,t]}(*)\right)^{sym}$ for $n > N$, $f_0(t) = g_0(t)$; the symbol "$*$" represents $n-2$ variables, $(g)^{sym}$ means the symmetrisation of the function with respect to all variables. Without loss of generality, for the representation of the abovementioned modification of the method it is sufficient to consider the case when $M_t = W_t$, $t \in [0,1]$ is a Wiener process, $G_t = g_0(t) + I_1(f_1(t))$: $x_{t,0} = g_0(t)$, and four first terms of the chaotic expansion are used. The relevant coefficient functions of the expansion in this case have the form

$$\begin{aligned}
&x_{t,0} = g_0(t), \qquad x_{t,1}(u_1) = g_{t,1}(u_1) + g_0(u_1)\alpha(u_1)1_{[0,t]}(u_1)\\
&x_{t,2}(u_2,u_3) = \left(g_1(u_2,u_3)\alpha(u_3)1_{[0,t]}(u_3) + g_0(u_2)\alpha(u_2)\alpha(u_3)1_{[0,u_3]}(u_2)1_{[0,t]}(u_3)\right)^{sym},\\
&x_{t,3}(u_1,u_2,u_3) = \big(g_1(u_1,u_2)\alpha(u_2)\alpha(u_3)1_{[0,u_3]}(u_2)1_{[0,t]}(u_3)+\\
&\qquad g_0(u_1)\alpha(u_1)\alpha(u_2)\alpha(u_3)1_{[0,u_2]}(u_1)1_{[0,u_3]}(u_2)1_{[0,t]}(u_3)\big)^{sym}.
\end{aligned}$$

Let us present the form of the approximating expression in Eq. (5) for the case of Wiener process, corresponding to the following integral in the expression for the third-order moment at $t_1 \le t_2 \le t_3$:

$$\int\limits_{[0,1]^3} x_{t_1,1}(u_1)x_{t_2,2}(u_2,u_3)x_{t_3,3}(u_1,u_2,u_3)du_1du_2du_3.$$

We introduce the notations $x_1(t) = x_{t,1}(u_1)$, $x_2(t) = x_{t,2}(u_2,u_3)$, $x_3(t) = x_{t,3}(u_1,u_2,u_3)$. Note that $x_{t,2}(u_2,u_3) = 0$ for $t < u_2 \wedge u_3$, and $x_{t,2}(u_2,u_3) \ne 0$ for $t \ge u_2 \wedge u_3$; $x_{t,3}(u_1,u_2,u_3) = 0$, when $t < u_1 \wedge u_2 \wedge u_3$, and $x_{t,3}(u_1,u_2,u_3) \ne 0$ for $t \ge u_1 \wedge u_2 \wedge u_3$. With this remark taken into account, the required modified approximating expression has the form

$$\begin{aligned}
&J_{1,2,3}(F) \equiv\\
&-\int\limits_{u_2\wedge u_3}^{1}\int\limits_{u_2\wedge u_3}^{1} \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right)\frac{\partial}{\partial \tau}\left(\frac{x_3(\tau)}{x_2(\tau)}\right)\right)^{\frac{1}{3}} 1_{[u_2\wedge u_3,\cdot]}(s)1_{[\cdot,1]}(\tau)\right)ds d\tau -\\
&\qquad \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)}\int\limits_{u_2\wedge u_3}^{1} \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_3(\tau)}{x_2(\tau)}\right)\right)^{1/3} 1_{[\cdot,1]}(\tau)\right)d\tau +\\
&\qquad \frac{x_3(1)}{x_2(1)}\int\limits_{u_2\wedge u_3}^{1} \Lambda F\left(c_k x_2(\cdot)\left(\frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right)\right)^{1/3} 1_{[u_2\wedge u_3,\cdot]}(s)\right)ds +\\
&\qquad\qquad \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)}\frac{x_3(1)}{x_2(1)}\Lambda F\left(c_k x_2(\cdot)\right).
\end{aligned}$$

Let us check the accuracy of approximation for the functional $F\left(y(\cdot)\right) = y(t_1)y(t_2)y(t_3)$:

$$J_{1,2,3}(F) = -c_k^3 \prod_{j=1}^{3} x_2(t_j) \int\limits_{u_2\wedge u_3}^{t_1\wedge t_2\wedge t_3} \frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right) ds \int\limits_{t_1\vee t_2\vee t_3}^{1} \frac{\partial}{\partial \tau}\left(\frac{x_3(\tau)}{x_2(\tau)}\right) d\tau -$$
$$\frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)} c_k^3 \prod_{j=1}^{3} x_2(t_j) \int\limits_{t_1\vee t_2\vee t_3}^{1} \frac{\partial}{\partial s}\left(\frac{x_3(\tau)}{x_2(\tau)}\right) d\tau +$$
$$\frac{x_3(1)}{x_2(1)} c_k^3 \prod_{j=1}^{3} x_2(t_j) \int\limits_{u_2\wedge u_3}^{t_1\wedge t_2\wedge t_3} \frac{\partial}{\partial s}\left(\frac{x_1(s)}{x_2(s)}\right) ds + \frac{x_1(0)}{x_2(0)}\frac{x_3(1)}{x_2(1)} c_k^3 \prod_{j=1}^{3} x_2(t_j) =$$
$$c_k^3 \prod_{j=1}^{3} x_2(t_j) \left(\frac{x_1(\bar{t_1})}{x_2(\bar{t_1})} - \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)}\right)\left(\frac{x_3(1)}{x_2(1)} - \frac{x_3(\bar{t_3})}{x_2(\bar{t_3})}\right) -$$
$$c_k^3 \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)} \prod_{j=1}^{3} x_2(t_j) \left(\frac{x_3(1)}{x_2(1)} - \frac{x_3(\bar{t_3})}{x_2(\bar{t_3})}\right) +$$
$$c_k^3 \frac{x_3(1)}{x_2(1)} \prod_{j=1}^{3} x_2(t_j) \left(\frac{x_1(\bar{t_1})}{x_2(\bar{t_1})} - \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)}\right) + c_k^3 \frac{x_1(u_2\wedge u_3)}{x_2(u_2\wedge u_3)}\frac{x_3(1)}{x_2(1)} \prod_{j=1}^{3} x_2(t_j) =$$
$$c_k^3 x_{t_1,1}(u_1) x_{t_2,2}(u_2,u_3) x_{t_3,3}(u_1,u_2,u_3).$$

Numerical results. Consider the numerical examples, illustrating the application of the derived formulae.

Example 1. Assume that in the expansion (1) $U = [0,1]\times[0,1]$, $dv(z) = \delta_{\{1\}}(z)$ is the Dirac measure at the point $z = 1$; $d\lambda(\tau,z) = d\tau$ on $\mathcal{B}([0,1]\times\{0\})$, $d\lambda(u) = d\lambda(t,z) = z^2\delta_{\{1\}}(z)dt = \delta_{\{1\}}(z)dt$ on $\mathcal{B}([0,1]^2\backslash(\tau,0))$. The Levy process in this case has the form $Y_t = M([0,t]\times(0,1]) = W_t + \int\limits_0^t \int\limits_{(0,1]} z d\tilde{N}(\tau,z) = W_t + \tilde{P}_t$, where $\tilde{P}_t = P_t - t$ is a centred Poisson process with the intensity 1. We calculate the mathematical expectation

$$E\left[F\left(X_{(\cdot)}\right)\right] = E\left[(1+\lambda X_t)^4\right], \tag{8}$$

where $X_t = I_1(x_{t,1}) + I_2(x_{t,2})$, $I_1(x_{t,1}) = \int\limits_U x_{t,1}(u)dM(u) = \int\limits_0^1 x_{t,1}(s)dW_s + \int\limits_0^1 x_{t,1}(s)d\tilde{P}_s$,

$$I_2(x_{t,2}) = \int\limits_{U^2} x_{t,2}(u_1,u_2)dM(u_1)dM(u_2) = \int\limits_0^1\int\limits_0^1 x_{t,2}(s_1,s_2)dW_{s_1}dW_{s_2} +$$

$$\int\limits_0^1\int\limits_0^1 x_{t,2}(s_1,s_2)dW_{s_1}d\tilde{P}_{s_2} + \int\limits_0^1\int\limits_0^1 x_{t,2}(s_1,s_2)d\tilde{P}_{s_1}dW_{s_2} + \int\limits_0^1\int\limits_0^1 x_{t,2}(s_1,s_2)d\tilde{P}_{s_1}d\tilde{P}_{s_2}.$$

As an example, we consider the functions $x_{t,1}(s) = f_t^2(s)$, $x_{t,2}(s_1,s_2) = f_t(s_1)f_t(s_2)$, for which the calculation of powers of multiple stochastic integrals reduces to the calculation of polynomials of the powers of single integrals, and one can find an exact value of the mathematical expectation (8). Indeed, in this case

$$X_t = I_1(x_{t,1}) + I_2(x_{t,2}) = \left(\int\limits_0^1 f^2(s)dW_s + \left(\int\limits_0^1 f^2(s)dW_s\right)^2 - \int\limits_0^1 f^2(s)ds\right) +$$
$$\left(\left(\int\limits_0^1 f(s)dP_s\right)^2 - \int\limits_0^1 f^2(s)ds\right) + 2\left(\int\limits_0^1 f(s)dW_s\right)\left(\int\limits_0^1 f(s)dP_s\right).$$

Using the formula

$$E\left[\left(\int\limits_0^1 g(s)dW_s\right)^n\right] = (2k-1)!!\left(\int\limits_0^1 g^2(s)ds\right)^k \quad \text{for} \quad n = 2k \tag{9}$$

(this expectation is zero for odd n) and the formula [7]

$$E\left[\left(\int\limits_0^T f_t(u)d\tilde{P}(u)\right)^{n+1}\right] = \sum_{k=0}^{n-1}\binom{n}{k}\int\limits_0^T f_t^{n-k+1}(u)du E\left[\left(\int\limits_0^T f_t(u)d\tilde{P}(u)\right)^{n+1}\right],$$

introducing the notation $I_2 = \int\limits_0^1 f^2(s)ds$, we arrive at the following exact values:

$$E\left[X_t^2\right] = 2I_4 + 8I_2^2, \quad E\left[X_t^3\right] = I_6 + 24I_4I_2 + 28I_3^2 + 64I_2^3,$$

$$E\left[X_t^4\right] = I_8 + 30I_6I_2 + 68I_5I_3 + 44I_4^2 + 516I_3^2I_2 + 357I_2^4 + 147I_4I_2^2,$$

and then the exact values of the expectation (8).

Table 1: Exact and approximate values for the function $f_t(s) = 1+ts$ for $A_1 = 1/3$, $A_2 = 1/6$, $c_1 = 1$, $c_2 = 2$, $\lambda = 4$!

t	0.1	0.3	0.5	0.7	0.9
Exact.	1.18521	1.29283	1.45708	1.7043	2.31586
Approx.	1.1725	1.26567	1.40156	1.59563	1.8676

Example 2. Assume that in Eq. (1) $Y_t = W_t$ i.e., consider a Wiener process. Let us calculate the mathematical expectation

$$E\left[F\left(X_{(\cdot)}\right)\right] = E\left[\exp\{\lambda X_t\}\right], \quad t \in [0,1], \quad 0 < \lambda < 1/2,$$

where $X_t = I_1(x_{t,1}) + I_2(x_{t,2})$, $I_1(x_{t,1}) = \int\limits_U x_{t,1}(u)dM(u) = \int\limits_0^1 x_{t,1}(s)dW_s$, $I_2(x_{t,2}) = \int\limits_{U^2} x_{t,2}(u_1,u_2)dM(u_1)dM(u_2) = \int\limits_0^1\int\limits_0^1 x_{t,2}(s_1,s_2)dW_{s_1}dW_{s_2}$, $x_{t,1}(s) = f_t^2(s)$, $x_{t,2}(s_1,s_2) = f_t(s_1)f_t(s_2)$. The exact value of this expectation is calculated using Eq. (9), because $I_2(x_{t,2}) = \int\limits_0^1\int\limits_0^1 f_t(s_1)f_t(s_2)dW_{s_1}dW_{s_2} = \left(\int\limits_0^1 f_t(s)dW_s\right)^2 - \int\limits_0^1 f_t^2(s)ds$, and is equal to

$$E\left[\exp\{\lambda X_t\}\right] = (1-2\lambda a)^{-1/2}\exp\left\{\frac{\lambda^2 a}{2(1-2\lambda a)} - \lambda a\right\}, \quad a = 1+t+t^2/3. \quad (10)$$

The expectation values for the function $f_t(s) = 1+ts$ calculated using Eq. (10) and the approximate Eq. (5) with the same values of the parameters A_1, A_2, c_1, c_2, as in the previous example, for $\lambda = 0.1$ are presented in Table 2.

Table 2: Exact and approximate values for the function $f_t(s) = 1+ts$ for $A_1 = 1/3$, $A_2 = 1/6$, $c_1 = 1$, $c_2 = 2$, $\lambda = 0.1$

t	0.1	0.3	0.5	0.7	0.9
Exact.	1.02163	1.03116	1.0446	1.0636	1.09062
Approx.	0.9966	1.01003	1.03057	1.0601	1.10097

The above numerical results demonstrate the accuracy of the approximation that agrees with the accuracy of the formulae exact for cubic functional polynomials in a given time interval, which allows using the derived approximate formulae in composite ones. The composite approximate formulae have been invented and widely used in the papers devoted to the computation of functional integrals (see [3, 4, 5]). The formulae of this structure, possessing the property of exactness for the functional polynomials of the fixed power, converge to the exact value of the integral in the specified class of integrable functionals. In the above references the composite formulae were obtained for functional integrals with the measures, generated by different classes of random processes (i.e., the mathematical expectations of the process realisations), including a wide class of processes with independent increments, as well as the solutions of some kinds of stochastic differential equations. From the computational point of view, the main feature of this type of composite formulae is the separation of the computation algorithm into two independently acting operators. The first one computes the mathematical expectations of the deterministic functions approximating the initial functional and depending on a finite number of random functionals with known mutual distributions. The second one provides the accuracy of the formula for the functional polynomials. Note that the construction of the composite formulae for mathematical expectations of functionals, defined by the expansions in multiple stochastic integrals requires the approximations of the multiple stochastic integrals on Levy processes. In the case of Wiener and Poisson processes, the corresponding approximations can be found, e.g., in [1, 2].

5. Conclusions

To calculate the mathematical expectation of random process functionals, representable by chaotic expansion in multiple stochastic Ito-Levy integrals we constructed an approximate formula that yields an exact result for functional polynomials of the process. We estimated the error of the derived approximate formula for the class of functionals that allow the representation as a sum of a cubic functional polynomial and a homogeneous polynomial of the fourth power with the bounded random coefficient function. The constructed formula is modified to calculate the mathematical expectation of functionals of the solution of linear stochastic equation with Skorokhod integral for the Wiener process with the initial condition in the form of a stochastic expansion with the fixed number of terms. Numerical examples illustrating the application of the derived formula are presented. The elaborated method gives a new useful tool for numerical integration with a required accuracy of stochastic differential equations which are reduced to the Skorokhod class of equations.

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