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On the crossing number of the join of the discrete graph with one graph of order five

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Abstract. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimal number of edge crossings over all drawings of G in the plane. In the paper, we extend results of the exact values of crossing numbers for join of graphs of order five. We give the crossing number of the join product $G + D_n$, where the graph G consists of one 4-cycle and one isolated vertex, and D_n consists on n isolated vertices.

Keywords: graph, drawing, crossing number, join product

MSC numbers: 05C10, 05C38

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1. Introduction

Let G be a simple graph with the vertex set V and the edge set E. A *drawing* of the graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. In such a drawing, the intersection of the interiors of the arcs is called a *crossing*. We assume that in a drawing no edge passes through any vertex other than its end-points, no two edges touch each other, and no three edges cross at the same point. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing. The *crossing number* $\mathrm{cr}(G)$ of a simple graph G is defined as the minimum possible number of edge crossings in a good drawing of G in the plane. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$.

Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $cr_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

In the paper, some proofs are based on the Kleitman's result on crossing numbers of the complete bipartite graphs [1]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

2. The crossing number of $G + D_n$

Let G be the graph consisting of one 4-cycle and of one isolated vertex. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where any vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G + D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

Name	Cyclic perm.	Name	Cyclic perm.	Name	Cyclic perm.
$P_1 \longrightarrow$	(1 2 3 4 5)	$P_9 \longrightarrow$	(1 2 5 3 4)	$P_{17} \longrightarrow$	(1 4 5 2 3)
$P_2 \longrightarrow$	(1 3 2 4 5)	$P_{10} \longrightarrow$	(1 5 2 3 4)	$P_{18} \longrightarrow$	(1 5 4 2 3)
$P_3 \longrightarrow$	(1 2 4 3 5)	$P_{11} \longrightarrow$	(1 5 3 2 4)	$P_{19} \longrightarrow$	(1 5 3 4 2)
$P_4 \longrightarrow$	(1 4 2 3 5)	$P_{12} \longrightarrow$	(1 3 5 2 4)	$P_{20} \longrightarrow$	(1 3 5 4 2)
$P_5 \longrightarrow$	(1 4 3 2 5)	$P_{13} \longrightarrow$	(1 2 5 4 3)	$P_{21} \longrightarrow$	(1 5 4 3 2)
$P_6 \longrightarrow$	(1 3 4 2 5)	$P_{14} \longrightarrow$	(1 5 2 4 3)	$P_{22} \longrightarrow$	(1 4 5 3 2)
$P_7 \longrightarrow$	(1 2 3 5 4)	$P_{15} \longrightarrow$	(1 2 4 5 3)	$P_{23} \longrightarrow$	(1 4 3 5 2)
$P_8 \longrightarrow$	(1 3 2 5 4)	$P_{16} \longrightarrow$	(1 4 2 5 3)	$P_{24} \longrightarrow$	(1 3 4 5 2)

Table 1: Names of Cyclic Permutations of 5-elements set

Let D be a good drawing of the graph $G+D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [3]. We emphasize that a rotation is a cyclic permutation. For $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, every subgraph $T^i \cup T^j$ of the graph $G + D_n$ is isomorphic with the graph $K_{5,2}$. In the paper, we will deal with the minimum necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced by the drawing D of the graph $G + D_n$ depending on the rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$.

D. R. Woodall [4] proved that, in any good drawing D of the graph $K_{5,2}$, $\mathrm{cr}_D(T^i, T^j) \geq 4$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$, then $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \leq \mathrm{cr}_D(T^i, T^j)$ and that $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \equiv \mathrm{cr}_D(T^i, T^j) (\mathrm{mod}\, 2)$.

We will separate the subgraphs T^i, $i = 1, \dots, n$, of the graph $G + D_n$ into three subsets depending on haw many the considered T^i crosses the edges of G in D. For $i = 1, 2, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, any F^i is exactly represented by $\mathrm{rot}_D(t_i)$. All cyclic permutations of five elements are collected in Table 1.

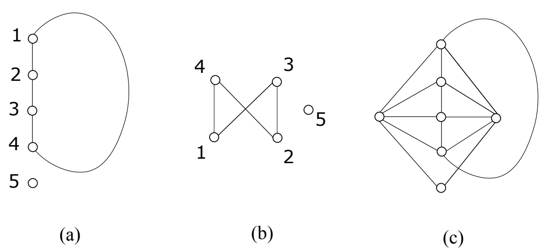

Figure 1: Two possible drawings of the graph G and the graph $G + D_2$

$\mathcal{M}_1$	$\mathcal{M}_2$
$A_1 : (15432)$	$A_2 : (14325)$
$A_3 : (14352)$	$A_4 : (14532)$

Table 2: Configurations of graph $G \cup T^i$ with vertices denoted of G as in Fig. 1(a)

$-$	A_1	A_2	A_3	A_4
A_1	4	3	2	3
A_2	3	4	3	2
A_3	2	3	4	3
A_4	3	2	3	4

Table 3: Lower-bounds of numbers of crossings for two configurations from $\mathcal{M}$

There is only one drawing of G without crossings shown in Figure 1(a). Assume a good drawing D of the graph $G + D_n$ in which the edges of G does not cross each other. We will count the number of necessary crossings between two subgraphs T^i and T^j with $\mathrm{cr}_D(G, T^i \cup T^j) = 0$. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Figure 1(a). It is easy to see that, in D, there are only four different possible configurations of F^i summarized in Table 2. We separate these configurations into two sets $\mathcal{M}_1 = \{A_1, A_3\}$ and $\mathcal{M}_2 = \{A_2, A_4\}$. We denote by $\mathcal{M}_D$ set of all configurations that exist in the drawing D belonging to the set $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. We denote by $\mathcal{M}_{1_D}$ and $\mathcal{M}_{2_D}$ sets of all configurations that exist in the drawing D belonging to the sets $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let X, Y be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(X, Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X, Y) = \min\{\mathrm{cr}_D(X, Y)\}$ over all good drawings of the graph $G + D_n$.

The configuration A_1 is represented by the cyclic permutation $P_{21} = (15432)$ and the configuration A_2 is represented by the cyclic permutation $P_5 = (14325)$. It was proved in [4] that there are necessary at least four interchanges of adjacent elements in P_{21} to obtain the inverse cyclic permutation $(12345) = P_1$. As P_5 is obtained from P_{21} by one interchange of elements 1 and 5, to obtain P_1 from P_5, at least three interchanges of adjacent elements are necessary. Thus, $\mathrm{cr}(A_1, A_2) \geq 3$. The same reason gives $\mathrm{cr}(A_1, A_4) \geq 3$, $\mathrm{cr}(A_2, A_3) \geq 3$, and $\mathrm{cr}(A_3, A_4) \geq 3$. Clearly, $\mathrm{cr}(A_i, A_i) \geq 4$ for all $i = 1, 2, 3, 4$. It is easy to verify that $\mathrm{cr}(A_1, A_3) \geq 2$ and also $\mathrm{cr}(A_2, A_4) \geq 2$. Thus, all lower-bounds of number of crossing of configurations from $\mathcal{M}$ are summarized in Table 3.

Similarly, there is only one drawing of G with one crossing among its edges. Assume now a good drawing D of the graph $G + D_n$ in which the edges of G cross once as shown in Figure 1(b). If we will deal with the vertex notation of G as shown in Figure 1(b), then, for the drawing D, we obtain the same set of configurations as in the previous case.

Lemma 1. *Let D be a good drawing of $G + D_n$, $n > 2$, in which* $\mathrm{cr}_D(T^i, T^j) \neq 0$ *for any different subgraphs T^i and T^j. Let $|R_D| > \left\lfloor \frac{n}{2} \right\rfloor$, $|S_D| < \left\lfloor \frac{n}{2} \right\rfloor$ and let $T^n, T^{n-1} \in R_D$ be different subgraphs. If both conditions*

$$\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 5 \qquad \textit{for any } T^i \in R_D \setminus \{T^n, T^{n-1}\}, \tag{2}$$

and

$$\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 4 \qquad \textit{for any } T^i \in S_D, \tag{3}$$

hold, or hods the condition

$$\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 6 \qquad \textit{for any } T^i \in R_D \setminus \{T^n, T^{n-1}\}, \tag{4}$$

then there are at least $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings in D.

Proof. Let $r = |R_D|$ and let $s = |S_D|$. So, the hypothesis implies that, $r \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $s + 1 \leq \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, by the assumption of lemma, $\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 4$ for any $T^i \notin R_D \cup S_D$, and the number of T^i that cross the graph G at least twice is $n - r - s$.

- We assume that both conditions (2) and (3) hold. Consequently

$$\begin{aligned}
\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(G \cup T^n \cup T^{n-1}) + \mathrm{cr}_D(K_{5,n-2}, G \cup T^n \cup T^{n-1}) \geq \\
&\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 5(r-2) + 4s + 4(n-r-s) + 1 = \\
&= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + r + 4n - 9 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1 + 4n - 9 = \\
&= 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

- We assume that the condition (4) holds. By the assumption of lemma, any $T^i \in S_D$ satisfies the condition $\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 3$. Consequently

$$\begin{aligned}
\mathrm{cr}_D(G + D_n) &\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(r-2) + 3s + 4(n-r-s) + 1 = \\
&= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2r - s + 4n - 11 \geq \\
&\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\Big(\left\lfloor \frac{n}{2} \right\rfloor + 1\Big) + 1 - \left\lfloor \frac{n}{2} \right\rfloor + 4n - 11 = \\
&= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 4n - 8 = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

□

Lemma 2. *Let D be a good drawing of $G + D_n$, $n > 2$, in which $\mathrm{cr}_D(T^i, T^j) \neq 0$ for any different subgraphs T^i and T^j. Let $\mathcal{M}_{1_D}$ and $\mathcal{M}_{2_D}$ be nonempty sets of configurations. If T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations from $\mathcal{M}_{1_D}$, $\mathcal{M}_{2_D}$, respectively, then*

$$\mathrm{cr}_D(T^n \cup T^{n-1}, T^i) \geq 3 \qquad \text{for any } T^i. \tag{5}$$

Proof. We need to show that there is no T^i with $\mathrm{cr}_D(T^n, T^i) = 1$ and $\mathrm{cr}_D(T^{n-1}, T^i) = 1$. For $\mathrm{cr}_D(T^n, T^i) = 1$, the inverse cyclic permutation to one which represents F^i must be obtained from the permutation which represent F^n by only one exchange of two adjacent elements. There are only five such permutations in Table 1. For $\mathrm{cr}_D(T^{n-1}, T^i) = 1$, the inverse cyclic permutation to one which represents F^i must be obtained from the permutation which represent F^{n-1} by only one exchange of two adjacent elements. But these five permutations are other than the previous five permutations. This completes the proof. □

Corollary 1. *Let D be a good drawing of $G + D_n$, $n > 2$, in which $\mathrm{cr}_D(T^i, T^j) \neq 0$ for any different subgraphs T^i and T^j. Let $\mathcal{M}_{1_D}$ and $\mathcal{M}_{2_D}$ be nonempty sets of configurations. If T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations from $\mathcal{M}_{1_D}$, $\mathcal{M}_{2_D}$, respectively, then*

$$\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 4 \qquad \text{for any } T^i \in S_D. \tag{6}$$

Proof. Clearly, because $\mathrm{cr}(G, T^i) = 1$ for $T^i \in S_D$. □

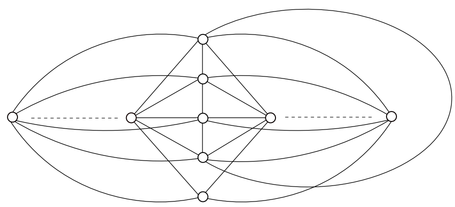

Figure 2: A good drawing of $G + D_n$

Theorem 1. $\mathrm{cr}(G + D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *for $n \geq 1$.*

Proof. In Figure 2 there is the drawing of $G + D_n$ with $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. We prove the reverse inequality by induction on n. The graph $G + D_1$ is planar, hence $\mathrm{cr}(G + D_1) = 0$. It is clear from Figure 1(c) that $\mathrm{cr}(G + D_2) \leq 1$. The graph $G + D_2$ contains a subdivision of K_5, and therefore $\mathrm{cr}(G + D_2) \geq 1$. So, $\mathrm{cr}(G + D_2) = 1$. So, the result is true for $n = 1$ and $n = 2$.

Suppose now that, for $n \geq 3$

$$\mathrm{cr}(G + D_{n-2}) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + \left\lfloor\frac{n-2}{2}\right\rfloor$$

and consider such a drawing D of $G + D_n$ that

$$\mathrm{cr}_D(G + D_n) < 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor. \tag{7}$$

The drawing D has the following property:

$$\mathrm{cr}_D(T^i, T^j) \neq 0 \qquad \text{for all } i, j = 1, 2, \dots, n,\ i \neq j. \tag{8}$$

To prove it assume that there are two different subgraphs T^i and T^j such that $\mathrm{cr}_D(T^i, T^j) = 0$ and let for every integer s, $s < n$, any good drawing of graph $G + D_s$ has at least $4\left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{s-1}{2}\right\rfloor + \left\lfloor\frac{s}{2}\right\rfloor$ crossings. Without loss of generality let $\mathrm{cr}_D(T^{n-1}, T^n) = 0$, one can easy to verify that $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 1$. By $\mathrm{cr}(K_{5,3}) = 4$ we give $\mathrm{cr}_D(T^k, T^{n-1} \cup T^n) \geq 4$ for $k = 1, 2, \dots, n-2$. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D\left(G \cup \bigcup_{i=1}^{n-2} T^i\right) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{5,n-2}, T^{n-1} \cup T^n) +$$

$$+\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + \left\lfloor\frac{n-2}{2}\right\rfloor + 4(n-2) + 1 =$$

$$= 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

This contradicts (7), and therefore $\mathrm{cr}_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2, \dots, n$, $i \neq j$. Our assumption on D together with $\mathrm{cr}(K_{5,n}) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{5,n}) < \left\lfloor\frac{n}{2}\right\rfloor.$$

Thus, we have $|R_D| > \left\lfloor\frac{n}{2}\right\rfloor$, $|S_D| < \left\lfloor\frac{n}{2}\right\rfloor$.

Since $\mathrm{cr}_D(G) \leq 1$, we will show that both conditions (2) and (3) hold, or the condition (4) holds using all above mentioned table in the following cases:

Case 1: $\mathrm{cr}_D(G) = 0$.

We will deal with the sets of configurations $\mathcal{M}_{1_D}$ and $\mathcal{M}_{2_D}$ in the drawing D.

1) $\mathcal{M}_{1_D} \neq \emptyset$ and $\mathcal{M}_{2_D} \neq \emptyset$.
   Without lost of generality if we fix any two T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations from $\mathcal{M}_{1_D}$, $\mathcal{M}_{2_D}$, respectively, then the condition (2) holds provided by only $\mathrm{cr}(A_1, A_3) = \mathrm{cr}(A_2, A_4) = 2$. The condition (3) is fulfilled by Corollary 1.

2) $\mathcal{M}_{1_D} = \emptyset$ or $\mathcal{M}_{2_D} = \emptyset$.

First, let us consider the case $\mathcal{M}_{1_D} = \emptyset$. Thus, $1 \leq |\mathcal{M}_{2_D}| \leq 2$. If $|\mathcal{M}_{2_D}| = 2$, then we fix any two T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have different configurations from $\mathcal{M}_{2_D}$, respectively. If $|\mathcal{M}_{2_D}| = 1$, then we fix any two different T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have the same configuration from $\mathcal{M}_{2_D}$. The condition (4) is fulfilled by Table 3 in the both cases. The similar idea is used for the case $\mathcal{M}_{2_D} = \emptyset$.

Case 2: $\mathrm{cr}_D(G) = 1$.
We will follow the same arguments for the sets of configurations $\mathcal{M}_{1_D}$ and $\mathcal{M}_{2_D}$ as in the previous case.

So, by Lemma 1, we obtain a contradiction with the assumption that there are less than $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings in the considered drawing D in all mentioned cases.

$\square$

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