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Circular orbits around static self-gravitating scalar field configurations

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Abstract. This article deals with innermost stable circular orbits (ISCOs) of neutral test particles around static, self-gravitating, spherically symmetric configurations that are formed and supported in equilibrium by a self-interacting real scalar field minimally coupled to gravity. For such objects, the spacetime metric can be expressed as the result of actions of some nonlinear integral operators (quadratures) on the scalar field. The quadratures do not depend on the form of the scalar field self-interaction potential. This feature enables us to classify the possible ISCOs, and to study the properties of various kinds of circular orbits without having to solve the Einstein-Klein-Gordon equations. It turns out that there exist two kinds of ISCOs, say the first and second kind, which are characterized, respectively, by a nonzero and zero specific angular momentum J of a test particle. We show that in the case of a classical scalar field, black holes have ISCOs only of the first kind. Naked singularities of general type have ISCOs only of the second kind, while some fine-tuning naked singularities and regular configurations have stable circular orbits of any positive radius. Black holes supported by phantom scalar fields can have ISCOs of both the kind depending on the Schwarzschild mass: for any one-parameter family of black holes parameterized by the mass, there exists a value $m_c > 0$ such that ISCOs of the first and second kind are in the intervals $m_c < m < \infty$ $(J > 0)$ and $0 < m \leqslant m_c$ $(J = 0)$, respectively. The orbital radius r_{ISCO} reaches its minimum value at $m = m_c$ and goes to infinity as m goes to zero or infinity.

Keywords: innermost stable circular orbit, scalar hair, black hole, wormhole, naked singularity, topological geon
MSC numbers: 83C10, 83C57

1. Introduction

Stable circular orbits play important roles both in theoretical and observational astrophysics [1, 2]. In particular, innermost stable circular orbits (ISCOs) around black holes are usually identified with inner edges of accretion disks. It seems to be quite adequate at least for thin disks, assuming that magnetohydrodynamic forces are weak in comparison with gravitational ones [3, 4]. From a theoretical point of view, as well as from an observational point of view, the radius of an ISCO and the orbital frequency (angular velocity) of a test particle are among the most informative parameters in the astrophysics of compact self-gravitating stellar objects. Circular orbits around vacuum black holes are well studied [5]. In the Universe, on the other hand, stellar-mass black holes and supermassive black holes in galactic nuclei are surrounded by dark matter and should not be thought of as objects immersed in an empty spacetime. One of the most notable models of dark matter is a self-interacting real scalar field [6, 7, 8] because it is thought to be involved only in interaction with gravity but not any other matter fields. Another compact objects of interest (at least to theorists), where scalar fields may naturally occur, are the so-called boson stars [9]; in the case of real fields, they are described mathematically by regular solutions of the Einstein-Klein-Gordon equations.

The question naturally arises whether one can distinguish stellar objects with scalar hair from other ones, simply by observing the behavior of test particles on circular orbits near the ISCO. It is not known at present whether a real scalar field exists in nature or we should consider it from a purely phenomenological point of view. In any case we also do not know the form of the self-interaction potential and there does not seem to be any guiding idea about its form in strong fields. On the one hand, there are uncountable number of degrees of freedom in the choice of the potential and it gives us efficient possibilities to model, in particular, the observed properties of dark matter, different kinds of boson stars, and scalar hair around compact self-gravitating objects. On the other hand, it requires that the problem for the Einstein-scalar field system be treated in some unified manner that would be appropriate, in some sense, for all admissible self-interaction potentials. Thus, it is reasonable to put the question in a different way: what can we say in general about ISCOs near scalar hairy objects?

The main goals of this article are to describe a general way of studying ISCOs around static, spherically symmetric compact objects with scalar hair, to classify the ISCOs, and to establish some of their basic properties without addressing the self-interaction potentials. It is possible because the circular orbits in a spherically symmetric spacetime are determined only by one of two independent metric functions. This function uniquely determines the corresponding effective potential of a test particle and the kind of an ISCO can be described — as will be seen below — in terms of a few parameters or another function belonging to a family of 'simple' functions (in the sense of their behaviour). We use the so-called 'inverse problem method for static scalar field configuration' [10, 11, 12, 13, 14, 15, 16] and the quadrature formulas obtained in Refs. [17, 18, 19]. This approach enables us to study the properties of circular orbits for arbitrary scalar field potentials without having to solve the Einstein-Klein-Gordon equations.

It is necessary to make some additional remarks concerning the stability of scalar field configurations and the so-called no-hair theorem [20]. First, it seems probable that most of scalar field configurations are unstable under linear perturbations. Second, the no-hair theorem are rules that forbid due to different reasons some hairy configurations; in our case, it says that if a static spherically symmetric configuration of a classical self-gravitating minimally coupled scalar field is an asymptotically flat black hole or a regular solution, then the field potential is negative in some neighbourhood of the horizon or, respectively, of the centre. Nevertheless, it is possible that these facts do not impose fatal restrictions on the domain of applicability of our results, because one can consider a scalar hairy stellar object without the event horizon. Then exterior metric remains the same as in the purely scalar field case and have to match with the corresponding interior solution. In addition, the object's surface should be located between the horizon and the photon sphere of the corresponding purely scalar field configuration, and the field potential should be positive up to the surface. Perhaps such scalar hairy objects would be more closely connected with the astronomical observations than the purely scalar field ones. Note, that there are many possibilities, which depend on the equation of state of ordinary matter, of extending of an exterior solution through the object's surface; in general, the interior solution has a very different behaviour than the extended purely scalar field solution. For example, a completely regular interior solution may have the exterior part which would be analytically extended to the centre as a naked singularity or a black hole if the solution was a purely scalar field one.

This article is organized as follows. Section 2 contains mathematical preliminaries. In Section 3 we consider spherically symmetric scalar field configurations. The quadrature formulas are written for two different coordinate systems, namely, for the the Schwarzschild-like coordinates and for the so-called quasiglobal coordinates. They are used, respectively, for the cases of classical and phantom scalar fields. Section 4 is devoted to exploring stable circular orbits in the general context of spacetime geometry near spherically symmetric scalar field configurations. In Section 5 we consider stable circular orbits around static, spherically symmetric scalar field configurations both for classical and phantom fields, that is, respectively, for canonical and noncanonical (negative in our notation) kinetic terms in the action. We prove some basic facts about ISCOs for classical scalar field configurations, including black holes and naked singularities, and describe an algorithm for studying circular orbits in the case of phantom fields. To demonstrate common features of circular orbits around phantom scalar field configuration, we explore in detail a one-parameter family of solutions, which includes wormholes, a topological geon, and black holes.

Throughout this article, we use the geometrical system of units with $G = c = 1$. Latin and Greek indices run from 0 to 3 and from 1 to 3, respectively. We adopt the metric signature $(+ - - -)$. Summation over any repeated index is assumed.

2. Local bases, connection and curvature

Geodesics are fully determined by geometry, therefore, it is useful first to outline the geometry of spherically symmetric spacetimes in a sufficiently general way. We will deal with the spherically symmetric metric

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{1}$$

where the metric functions A, B, and C depend only on the coordinates t and r. For the sake of generality we retain the ambiguity in the choice of the radial coordinate, so that in each specific case we can apply an appropriate gauge condition to the metric functions.

It is convenient to use the orthonormal basis of vector fields, associated with the metric (1), and the dual basis of 1-forms, so that the metric components become $(g_{ij}) = \mathrm{diag}\{1, -1, -1, -1\}$. These bases are given, respectively, by

$$\mathrm{e}_0 = \frac{1}{A}\,\partial_t, \quad \mathrm{e}_1 = \frac{1}{B}\,\partial_r, \quad \mathrm{e}_2 = \frac{1}{C}\,\partial_\theta, \quad \mathrm{e}_3 = \frac{1}{C\sin\theta}\,\partial_\varphi, \tag{2}$$

and

$$\mathrm{e}^0 = A\,dt\,, \quad \mathrm{e}^1 = B\,dr\,, \quad \mathrm{e}^2 = C\,d\theta\,, \quad \mathrm{e}^3 = C\sin\theta\,d\varphi. \tag{3}$$

We need also the corresponding orthonormal basis of 2-forms

$$\alpha^1 = \mathrm{e}^0\wedge\mathrm{e}^1,\ \alpha^2 = \mathrm{e}^0\wedge\mathrm{e}^2,\ \alpha^3 = \mathrm{e}^0\wedge\mathrm{e}^3,\ *\alpha^1 = \mathrm{e}^3\wedge\mathrm{e}^2,\ *\alpha^2 = \mathrm{e}^1\wedge\mathrm{e}^3,\ *\alpha^3 = \mathrm{e}^2\wedge\mathrm{e}^1, \tag{4}$$

where $*$ is the Hodge star operator. Furthermore, from here some convenient notation will be used: the directional derivatives along the basis vector fields (2) will be denoted by the corresponding subscript indices placed in the opposite order in parentheses (that can be omitted in practical calculations). For example,

$$\mathrm{e}_0\phi \equiv \phi_{(0)} = \frac{1}{A}\,\partial_t\phi\,, \quad \mathrm{e}_0\mathrm{e}_1 C \equiv C_{(1)(0)} = \frac{1}{A}\,\partial_t\left(\frac{1}{B}\,\partial_r C\right).$$

The connection 1-forms in the bases (2) and (3) are defined by the usual rule $\nabla_X e_j = \omega^i_j(X)\,\mathrm{e}_i$ and can be obtained by calculating the exterior derivative $d\mathrm{e}^i$ and applying Cartan's first structure equation [5]

$$d\mathrm{e}^i + \omega^i_j \wedge \mathrm{e}^j = 0\,.$$

Calculations of the curvature consist in applying the Cartan's second structure equation

$$\frac{1}{2}\,R_{ijkl}\,\mathrm{e}^k\wedge\mathrm{e}^l = g_{im}\,(d\omega^m_j + \omega^m_p\wedge\omega^p_j)\,.$$

Without going into details, the algebraically independent connection 1-forms and the curvature components are

$$\omega^0_1 = \frac{A_{(1)}}{A}\,\mathrm{e}^0 + \frac{B_{(0)}}{B}\,\mathrm{e}^1, \quad \omega^0_2 = \frac{C_{(0)}}{C}\,\mathrm{e}^2, \quad \omega^0_3 = \frac{C_{(0)}}{C}\,\mathrm{e}^3, \quad \omega^0_\alpha = \omega^\alpha_0,$$
$$\omega^1_2 = -\frac{C_{(1)}}{C}\,\mathrm{e}^2\,, \quad \omega^1_3 = -\frac{C_{(1)}}{C}\,\mathrm{e}^3\,, \quad \omega^2_3 = -\frac{\cot\theta}{C}\,\mathrm{e}^3\,, \quad \omega^\alpha_\beta = -\omega^\beta_\alpha\,, \tag{5}$$

$$
\begin{aligned}
R_{0101} = \frac{B_{(0)(0)}}{B} - \frac{A_{(1)(1)}}{A}, \quad & R_{0202} = R_{0303} = \frac{C_{(0)(0)}}{C} - \frac{A_{(1)}C_{(1)}}{AC}, \\
R_{0212} = R_{0313} &= \frac{C_{(1)(0)}}{C} - \frac{A_{(1)}C_{(0)}}{AC}, \\
R_{1212} = R_{1313} = \frac{C_{(1)(1)}}{C} &- \frac{B_{(0)}C_{(0)}}{BC}, \quad R_{2323} = \frac{C^2_{(1)} - C^2_{(0)} - 1}{C^2}.
\end{aligned} \tag{6}
$$

It is useful to write the curvature as

$$
\begin{aligned}
R = &\left(\frac{B_{(0)(0)}}{B} - \frac{A_{(1)(1)}}{A}\right)\alpha^1 \otimes \alpha^1 + \left(\frac{C_{(0)(0)}}{C} - \frac{A_{(1)}C_{(1)}}{AC}\right)(\alpha^2 \otimes \alpha^2 + \alpha^3 \otimes \alpha^3) \\
&+ \frac{C^2_{(1)} - C^2_{(0)} - 1}{C^2} *\alpha^1 \otimes *\alpha^1 + \left(\frac{C_{(1)(1)}}{C} - \frac{B_{(0)}C_{(0)}}{BC}\right)(*\alpha^2 \otimes *\alpha^2 + *\alpha^3 \otimes *\alpha^3) \\
&+ \left(\frac{C_{(1)(0)}}{C} - \frac{A_{(1)}C_{(0)}}{AC}\right)(\alpha^3 \otimes *\alpha^2 + *\alpha^2 \otimes \alpha^3 - \alpha^2 \otimes *\alpha^3 - *\alpha^3 \otimes \alpha^2),
\end{aligned}
$$

in order to show its structure in spherically symmetric spacetimes more explicitly. In calculating the curvature we take into account the identity

$$
\frac{C_{(1)(0)}}{C} - \frac{A_{(1)}\,C_{(0)}}{A\,C} = \frac{C_{(0)(1)}}{C} - \frac{B_{(0)}C_{(1)}}{B\,C}\,.
$$

For the sake of completeness, it is also useful to write out nonzero components of the Ricci tensor $\mathcal{R}$ and the scalar curvature S. They are

$$
\begin{aligned}
\mathcal{R}_{00} = \frac{A_{(1)(1)}}{A} - \frac{B_{(0)(0)}}{B} - 2\,\frac{C_{(0)(0)}}{C} + 2\,\frac{A_{(1)}C_{(1)}}{AC}, \quad & \mathcal{R}_{01} = 2\,\frac{A_{(1)}C_{(0)}}{AC} - 2\,\frac{C_{(1)(0)}}{C}, \\
\mathcal{R}_{11} = \frac{B_{(0)(0)}}{B} - \frac{A_{(1)(1)}}{A} - 2\,\frac{C_{(1)(1)}}{C} + 2\,\frac{B_{(0)}C_{(0)}}{BC}&, \\
\mathcal{R}_{22} = \mathcal{R}_{33} = \frac{C_{(0)(0)}}{C} - \frac{A_{(1)}C_{(1)}}{AC} + \frac{B_{(0)}C_{(0)}}{BC} &- \frac{C_{(1)(1)}}{C} - \frac{C^2_{(1)} - C^2_{(0)} - 1}{C^2},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{S}{2} = \frac{A_{(1)(1)}}{A} - \frac{B_{(0)(0)}}{B} - 2\,\frac{C_{(0)(0)}}{C} &+ 2\,\frac{A_{(1)}C_{(1)}}{AC} \\
&+ 2\,\frac{C_{(1)(1)}}{C} - 2\,\frac{B_{(0)}C_{(0)}}{BC} + \frac{C^2_{(1)} - C^2_{(0)} - 1}{C^2}.
\end{aligned} \tag{8}
$$

3. Self-gravitating scalar field configurations

The action with minimal coupling between curvature and a real scalar field ϕ has the form (in geometric units, $G = c = 1$)

$$
\Sigma = \frac{1}{8\pi}\int\left(-\frac{1}{2}S + \varepsilon\langle d\phi, d\phi\rangle - 2V(\phi)\right)\sqrt{|g|}\,d^4x\,,
$$

where $\varepsilon = \pm 1$, $V(\phi)$ is a self-interaction potential, and the angle brackets denote the scalar product with respect to the metric. The energy-momentum tensor has the form

$$8\pi \mathcal{T} \;=\; 2\,\varepsilon\, d\phi \otimes d\phi \;-\; (\varepsilon\,\langle d\phi, d\phi\rangle - 2V)\, g\,. \tag{9}$$

Assuming $\phi = \phi(t, r)$ and using (7), (8), and (9), the Einstein-Klein-Gordon system for a spherically symmetric spacetime with the metric (1) can be written as

$$\{00\}: \quad -2\frac{C_{(1)(1)}}{C} + 2\frac{B_{(0)}C_{(0)}}{BC} - \frac{C_{(1)}^2 - C_{(0)}^2 - 1}{C^2} = \varepsilon(\phi_{(0)}^2 + \phi_{(1)}^2) + 2V\,, \tag{10}$$

$$\{11\}: \quad -2\frac{C_{(0)(0)}}{C} + 2\frac{A_{(1)}C_{(1)}}{AC} + \frac{C_{(1)}^2 - C_{(0)}^2 - 1}{C^2} = \varepsilon(\phi_{(0)}^2 + \phi_{(1)}^2) - 2V\,, \tag{11}$$

$$\begin{aligned}\{22\}: \quad \frac{A_{(1)(1)}}{A} - \frac{B_{(0)(0)}}{B} + \frac{C_{(1)(1)}}{C} - \frac{C_{(0)(0)}}{C} + \frac{A_{(1)}C_{(1)}}{AC} - \frac{B_{(0)}C_{(0)}}{BC}& \\ = \varepsilon(\phi_{(0)}^2 - \phi_{(1)}^2) - 2V\,,&\end{aligned} \tag{12}$$

$$\{01\}: \quad -2\frac{C_{(0)(1)}}{C} + 2\frac{B_{(0)}C_{(1)}}{BC} \;=\; 2\,\varepsilon\phi_{(0)}\phi_{(1)}\,, \tag{13}$$

$$\Box\phi + \varepsilon V_\phi' = 0: \quad \phi_{(0)(0)} - \phi_{(1)(1)} + \phi_{(0)}\frac{(BC^2)_{(0)}}{BC^2} - \phi_{(1)}\frac{(AC^2)_{(1)}}{AC^2} + \varepsilon V_\phi' \;=\; 0\,. \tag{14}$$

In the case of static spacetimes the system (10) — (14) can be reduced to the one involving quadratures only [17, 18, 19]. Such a reduction is possible for both the basic gauge conditions in the metric (1), namely, $C = r$ and $B = 1/A$.

The gauge condition $C = r$ (r is the area coordinate) defines *the Schwarzschild-like coordinates*, and will be used below in our analysis of circular orbits around black holes, regular configurations, and naked singularities supported by classical scalar fields. It is convenient to rewrite the metric (1) in the form

$$ds^2 = \mathrm{e}^{2F} f dt^2 - \frac{dr^2}{f} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (\text{so that } \ A^2 = \mathrm{e}^{2F} f,\ B^2 = 1/f). \tag{15}$$

In this case each strictly monotonic function $\phi(r)$ of class $\mathcal{C}^2$, with the asymptotic behaviour $\phi = O\big(r^{-1/2-\,\alpha}\big)$ $(\alpha > 0)$, determines a one-parameter family of solutions to the system (10) — (14) by the quadratures [19]

$$F(r) = -\,\varepsilon\int_r^\infty \phi'^2 r dr\,, \quad \xi(r) = r + \int_r^\infty \big(1 - \mathrm{e}^F\big)dr\,, \tag{16}$$

$$A^2 = 2r^2\int_r^\infty \frac{\xi - 3m}{r^4}\,\mathrm{e}^F dr\,, \quad f(r) = \mathrm{e}^{-2F}A^2\,, \tag{17}$$

$$\widetilde{V}(r) = \frac{1}{2r^2}\left(1 - 3f + \varepsilon r^2 \phi'^2 f + 2\,\mathrm{e}^{-F}\,\frac{\xi - 3m}{r}\right), \tag{18}$$

where the parameter m (Schwarzschild mass) takes arbitrary real values.

These formulas give a general solution to the 'inverse problem' for spherically symmetric self-gravitating scalar field configurations formulated in [15, 16]. It means that for a given monotonic $\phi(r)$, one sequentially finds the functions e^F, ξ, A^2, f, $\widetilde{V}(r)$, and then find the potential $V(\phi) = \widetilde{V}(r(\phi))$. This mathematical technique allows us to examine the problem for large classes of admissible self-interaction potentials. It is important because we have no a priori knowledge of the form of the potential. Since the functions ϕ, e^F, and ξ uniquely determine each other, one can start by specifying either the function e^F or ξ instead of ϕ. Note also that each solution to the system (10) — (14) satisfies (16) — (18) regardless of the monotonicity of ϕ.

It turns out that for black holes, wormholes, and topological geons supported by phantom scalar fields ($\varepsilon = -1$), the gauge condition $B = 1/A$ is more applicable. It defines the so-called *quasiglobal coordinates* [21] in which the metric and the quadratures can be written as [18, 19]

$$ds^2 = A^2 dt^2 - \frac{dr^2}{A^2} - C^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{19}$$

$$\phi' = \sqrt{-\varepsilon C''/C}\,, \qquad A^2 = 2C^2 \int\limits_r^\infty \frac{r - 3m}{C^4}\, dr\,, \tag{20}$$

$$\widetilde{V}(r) = \frac{1}{2C^2}\left(1 - 3C'^2 A^2 - CC''A^2 + 2C'\,\frac{r - 3m}{C}\right), \tag{21}$$

where the mass m takes arbitrary real values.[1]

The solution (19) — (21) is applicable in the general case. In order to use these formulas for phantom scalar fields, it is necessary to specify a monotonically increasing function $C(r)$ satisfying the condition $C'' \geqslant 0$ (for $\varepsilon = -1$) on $\mathbb{R}_+$ and having the asymptotic behaviour

$$C = r + o(1) \quad \text{as} \quad r \to \infty\,. \tag{22}$$

The condition (22) guarantees that m will have sense of the Schwarzschild mass. Then the field function $\phi(r)$, the metric function A^2, and the function $\widetilde{V}(r)$ can be found from (20) and (21) by direct calculation. In the case of $\varepsilon = 1$, one should choose $C(r)$ such that $C'' \leqslant 0$.

[1] In our notation r and $C(r)$ correspond, respectively, to ξ and $r(\xi)$ in the notation of [19].

4. Circular orbits in spherically symmetric spacetimes

4.1 The geodesics equations

The geodesic equation $\nabla_U U = 0$ in the basis (2) gives the four equations

$$\frac{dU^i}{ds} + \omega^i_j(U)\, U^j = 0 \tag{23}$$

for the components of the four-velocity

$$U = U^0 \mathrm{e}_0 + U^1 \mathrm{e}_1 + U^2 \mathrm{e}_2 + U^3 \mathrm{e}_3,$$

where

$$U^0 = A\frac{dt}{ds}\,, \quad U^1 = B\frac{dr}{ds}\,, \quad U^2 = C\frac{d\theta}{ds}\,, \quad U^3 = C\sin\theta\frac{d\varphi}{ds}\,.$$

In spherically symmetric spacetimes, without loss of generality, we may assume that a geodesic has the initial conditions

$$U^2 = 0\,,\ \theta = \pi/2\,. \tag{24}$$

Then, we see from the 2-component of Eq. (23),

$$\frac{dU^2}{ds} + \frac{C_{(0)}}{C}\,U^2U^0 + \frac{C_{(1)}}{C}\,U^2U^1 - \frac{\cot\theta}{C}\,U^3U^3 = 0\,,$$

that the geodesic under consideration are entirely in the equatorial plane. In what follows, we suppose that the conditions (24) are fulfilled.

The remaining geodesic equations take the form

$$\frac{dU^0}{ds} + \frac{A_{(1)}}{A}\,U^0U^1 + \frac{B_{(0)}}{B}\,U^1U^1 + \frac{C_{(0)}}{C}\,U^3U^3 = 0\,, \tag{25}$$

$$\frac{dU^1}{ds} + \frac{A_{(1)}}{A}\,U^0U^0 + \frac{B_{(0)}}{B}\,U^0U^1 - \frac{C_{(1)}}{C}\,U^3U^3 = 0\,, \tag{26}$$

$$\frac{dU^3}{ds} + \frac{C_{(0)}}{C}\,U^0U^3 + \frac{C_{(1)}}{C}\,U^1U^3 = 0\,. \tag{27}$$

Subtracting Eqs. (26) and (27), multiplied, respectively, by U^1 and U^3, from Eq. (25) multiplied by U^0, one obtains the obvious first integral of motion

$$\left(U^0\right)^2 - \left(U^1\right)^2 - \left(U^3\right)^2 = \mathrm{k}\,, \quad \mathrm{k} = -1\,,\, 0\,,\, 1, \tag{28}$$

where $\mathrm{k} = -1$, $\mathrm{k} = 0$, and $\mathrm{k} = 1$ for spacelike, null, and timelike geodesics respectively. Another first integral of motion (specific angular momentum of a test particle),

$$U^3C = J, \quad \text{or} \quad C^2\frac{d\varphi}{ds} = J, \quad (J = \text{constant}), \tag{29}$$

can be deduced from Eq. (27) by taking into account the equality

$$C_{(0)}U^0 + C_{(1)}U^1 = dC/ds$$

which holds along the geodesic.

In *static* spacetimes, one more first integral of motion (specific energy of a test particle) follows, in an analogous manner, from Eq. (25). One has

$$U^0 A = E, \quad \text{or} \quad A^2 \frac{dt}{ds} = E \quad (E = \text{constant}), \tag{30}$$

so that the first integral (28) can be rewritten, by substituting for U^3 and U^0 from (29) and (30), in the form

$$\left(\frac{dr}{ds}\right)^2 = \frac{1}{A^2 B^2}\left(E^2 - V_{eff}\right), \tag{31}$$

where the effective potential is defined by

$$V_{eff} = A^2\left(\mathrm{k} + \frac{J^2}{C^2}\right). \tag{32}$$

4.2 Circular orbits

From now on we will restrict our attention to null ($k = 0$) and timelike ($k = 1$) circular orbits in a static, asymptotically flat, spherically symmetric spacetime with a nonnegative mass m. The latter restriction is reasonable since for $\varepsilon = 1$ the condition $m \leqslant 0$ determines only naked singularities which have no circular orbits at all; for $\varepsilon = -1$ it is sufficient to consider only the case $m \geqslant 0$, because the change $m \to -m$ gives us the solution which may be obtained by mirror-reflecting through the hypersurface $r = 0$. We also will suppose that the orbit under consideration corresponds to a test particle that is 'visible' to distant observers; in particular, the orbit is placed outside the event horizon in black hole spacetimes and outside the throat in wormholes.

Since $U^1 = 0$ in a circular orbit, we obtain from (26) and (28) — (30) the equations

$$\frac{A'}{A}U^0U^0 - \frac{C'}{C}U^3U^3 = \frac{A'}{A}\frac{E^2}{A^2} - \frac{C'}{C}\frac{J^2}{C^2} = 0, \quad \left(U^0\right)^2 - \left(U^3\right)^2 = \mathrm{k}, \tag{33}$$

which for massive particles give

$$E^2 = \frac{A^3C'}{AC' - A'C}, \quad J^2 = \frac{C^3A'}{AC' - A'C}, \quad U^0 = \frac{E}{A}, \quad U^3 = \frac{J}{C}. \tag{34}$$

Using these formulas for a given orbital radius (i. e. a value of the radial coordinate), one can straightforwardly calculate the specific energy, the specific angular momentum, and the angular velocity (orbital frequency) of a test particle. It is clear that circular orbits exist only in the region where the condition

$$AC' - A'C \geqslant 0 \tag{35}$$

holds; here the equality sign corresponds to massless test particles ($\mathrm{k} = 0$) moving on the so-called *photon orbit* of some radius r_{ph}. It is a *necessary* but not sufficient condition: for instance, it holds in any flat spacetime, but there are no circular orbits. In order to obtain a convenient sufficient condition, one needs to use the effective potential.

The first derivative of the effective potential (32),

$$V'_{eff} = \mathrm{k}\left(A^2\right)' + \left(\left(A^2\right)' - 2A^2\frac{C'}{C}\right)\frac{J^2}{C^2}, \tag{36}$$

can be rewritten, using the last equation in (33) and the last two equations in (34), in the form

$$V'_{eff} = 2A^2\left(\frac{A'}{A}\frac{E^2}{A^2} - \frac{C'}{C}\frac{J^2}{C^2}\right),$$

and hence (taking into account (33)) equations determining circular orbits become

$$V'_{eff} = 0\,, \quad E^2 = A^2\left(\mathrm{k} + \frac{J^2}{C^2}\right), \quad U^0 = \frac{E}{A}, \quad U^3 = \frac{J}{C}\,. \tag{37}$$

The subsequent arguments are quite similar to the ones known for the Keplerian orbits in classical mechanics and for bound orbits in the Schwarzschild spacetime. The result is that each local extremum r of the effective potential determines a circular orbit of the same radius: namely, the orbit is stable or unstable if r is minimum or a maximum, respectively. The stable orbits obey the condition $V''_{eff} > 0$.

The orbit at some $r = r_{ISCO} > r_{ph}$ is said to be an *innermost stable circular orbit* (ISCO) if there are no stable orbits with radiuses smaller than r_{ISCO} and there exist stable orbits for each $r > r_{ISCO}$. As a rule the ISCO occurs when J decreases and the leftmost minimum of the effective potential degenerates into an inflection point. For example, as we will see below in this section, for a black hole supported by a scalar field with positive kinetic energy, there exists some constant $J_{ISCO} > 0$, depending on the forms of A and C, such that the first equation in (37) has at least two solutions $r(J)$ for each specific angular momentum in the interval (J_{ISCO}, ∞), has a unique solution at $J = J_{ISCO}$, and has no solutions in the interval $(0, J_{ISCO})$. It is obvious that $r(J_{ISCO}) = r_{ISCO}$. Below the ISCOs with $J_{ISCO} > 0$ and $J_{ISCO} = 0$ will be referred to as the ones of *the first and the second kind* respectively.

The inflection point of the effective potential obeys the system of equations

$$V'_{eff} = 0\,, \quad V''_{eff} = 0\,. \tag{38}$$

In Ref. [22, 23] the second equation is written only in terms of the derivatives of the metric function up to the second order, using a different form of the effective potential, namely, $V_{eff}(r,E,J) = E^2/A^2 - J^2/C^2 - 1$. We prefer to use the effective potential $V_{eff}(r,J)$ in the form (32), in which it depends only on J, but not on E: it is not essential for the final results but makes the subsequent analysis much easier. For the sake of completeness, here we derive in our notation a convenient explicit form of the system (38) (note that it makes sense only for massive particles, that is, in the case $\mathrm{k}=1$).

At extremum points of the effective potential, its second derivative with respect to r can be expressed in the form

$$\begin{aligned} V''_{eff}\Big|_{V'_{eff}=0} &= \left(\frac{(A^2)'}{A^2}\right)' V_{eff} - 2\left((A^2)'\frac{C'}{C} + A^2\frac{C''}{C} - 3A^2\frac{C'^2}{C^2}\right)\frac{J^2}{C^2} \\ &= A^2\left\{\left(\frac{(A^2)'}{A^2}\right)' + \left(\left(\frac{(A^2)'}{A^2}\right)' - 2\frac{(A^2)'}{A^2}\frac{C'}{C} - 2\frac{C''}{C} + 6\frac{C'^2}{C^2}\right)\frac{J^2}{C^2}\right\}. \end{aligned}$$

Therefore, the system (38) (divided by A^2) is equivalent to the system

$$\frac{(A^2)'}{A^2} + \left(\frac{(A^2)'}{A^2} - 2\frac{C'}{C}\right)\frac{J^2}{C^2} = 0\,, \tag{39}$$

$$\left(\frac{(A^2)'}{A^2}\right)' + \left(\left(\frac{(A^2)'}{A^2}\right)' - 2\frac{(A^2)'}{A^2}\frac{C'}{C} - 2\frac{C''}{C} + 6\frac{C'^2}{C^2}\right)\frac{J^2}{C^2} = 0\,. \tag{40}$$

It follows from the condition (35) that the coefficient of J^2/C^2 in the second term of Eq. (39) is negative in the region, where circular orbits exist. Eliminating J^2/C^2 from Eqs. (39) and (40), we obtain

$$\left(\frac{(A^2)'}{A^2}\right)' - \left(\frac{(A^2)'}{A^2}\right)^2 - \frac{C''}{C'}\frac{(A^2)'}{A^2} + 3\frac{C'}{C}\frac{(A^2)'}{A^2} = 0 \quad \Leftrightarrow \quad \left\{\ln\left(\frac{A'C^3}{A^3C'}\right)\right\}' = 0\,.$$

Finally, Eqs. (38), which determine r_{ISCO} and J_{ISCO}, take the form

$$\left(\frac{A'C^3}{A^3C'}\right)' = 0\,, \qquad J^2_{ISCO} = \left.\frac{A'C^3}{AC' - A'C}\right|_{r\,=\,r_{ISCO}}. \tag{41}$$

In order to obtain the ISCO parameters, one should find, if any, solutions of the first equation in the region $r > r_{ph}$, and then choose (among them) the leftmost solution for which the second equation gives a positive value of J^2. Equations (37) enable one to find the specific energy and the angular velocity.

For the angular velocity $\omega = d\varphi/dt$ of a test particle as measured by a static observer at infinity, from the last two equations in (37) we find

$$\omega^2 = \frac{J^2}{E^2}\frac{A^4}{C^4}\,. \tag{42}$$

It is important that the angular velocity of a test particle at a circular orbit depends only on its radius and can be expressed only in terms of the metric functions. From (42), taking (34) into account, we obtain

$$\omega = \sqrt{(A^2)'/(C^2)'}\,. \tag{43}$$

5. Circular orbits around scalar field configurations

5.1 Scalar field configurations with $\varepsilon = 1$

First we consider the case of *black holes* with classical scalar field hair, for which $\xi(0) < 3m$. In the Schwarzschild-like coordinates (see Eqs. (15) — (18))

$$AC' - A'C = \frac{1}{A}\left(A^2C' - \frac{1}{2}\left(A^2\right)'C\right) = \frac{\xi - 3m}{rA}\,\mathrm{e}^F\,,$$

so that the necessary condition (35) for the existence of circular orbits becomes

$$\xi - 3m \geqslant 0\,. \tag{44}$$

Let r_{ph} be the (unique) solution of the equation $\xi(r) - 3m = 0$, that is, the radius of the photon sphere. The condition (44) holds for all $r > r_{ph}$ because $\xi(r)$ is a monotonically increasing function (see Eq. (16) and the right panel in Figure 1).

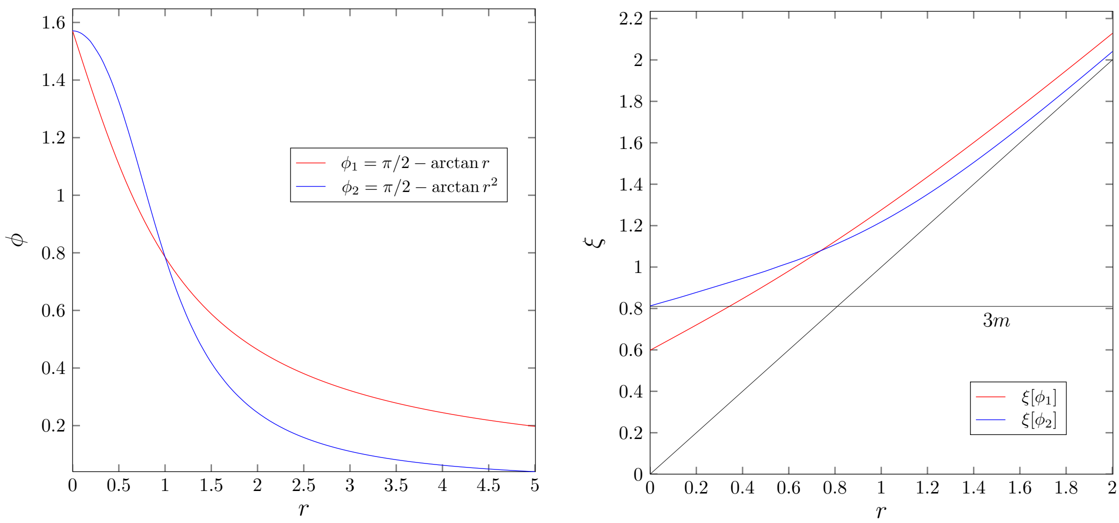

Figure 1: Two examples of families of solutions with self-gravitating scalar hair. Each of these families is completely determined by the corresponding scalar field. The map $\xi[\phi]$, which is defined by (16), depends only on the form of $\phi(r)$; its value at any specified field is simply a function with the asymptotic behaviour $\xi(r) = r + o(1)$ as $r \to \infty$. The Schwarzschild mass m determines the type of the solutions. For $\phi_1(r)$ and $3m > \xi[\phi_1](0) = 0.594$, the spacetime is a black hole, while for $3m < \xi[\phi_1](0)$ it is a naked singularity; the equality sign also corresponds to a naked singularity, because of $\phi_1 = \pi/2 - r + o(r)$ as $r \to 0$. For $\phi_2(r)$, the situation differs from the previous one only in that $\phi_2 = \pi/2 - r^2 + o(r^2)$ as $r \to 0$, and hence the equality $3m = \xi[\phi_2](0) = 0.813$ corresponds to a regular configuration.

Now we need the following two lemmas.

Lemma 1. *Let $\varepsilon = 1$. If $\xi(0) < 3m$ (black holes) then $\mathrm{e}^F > \dfrac{\xi - 3m}{r}$ for all $r > 0$.*

Since

$$\mathrm{e}^F = 1 + o(1/r), \quad r \to \infty,$$

the integration by parts in (16) yields

$$\xi(r) = r + \int\limits_r^\infty (1 - \mathrm{e}^F)\, dr = r\mathrm{e}^F + \int\limits_r^\infty \left(\mathrm{e}^F\right)' r dr\,.$$

If $\varepsilon = 1$ then $\left(\mathrm{e}^F\right)' \geqslant 0$ for all $r > 0$, and therefore

$$r\mathrm{e}^F - \xi(r) + 3m = 3m - \int\limits_r^\infty \left(\mathrm{e}^F\right)' r dr \;>$$

$$\xi(0) - \int\limits_r^\infty \left(\mathrm{e}^F\right)' r dr = \int\limits_0^\infty \left(\mathrm{e}^F\right)' r dr - \int\limits_r^\infty \left(\mathrm{e}^F\right)' r dr = \int\limits_0^r \left(\mathrm{e}^F\right)' r dr \geqslant 0\,.$$

This proves the lemma. □

Lemma 2. *Let $\varepsilon = 1$ and $m > 0$. If the metric (15) describes (in the Schwarzschild-like coordinates) the spacetime around a black hole ($\xi(0) < 3m$), then $\left(A^2\right)' > 0$ in the region $[r_{ph}, \infty)$.*

In this case, the metric has the form (15). From (17), the derivative of A^2 with respect to r can be written in the form

$$\left(A^2\right)' = \frac{2}{r}\left(A^2 - \frac{\xi - 3m}{r}\mathrm{e}^F\right).$$

Integrating by parts in (17) and applying the identity $(\xi - 3m)' = \mathrm{e}^F$, we obtain

$$A^2 = 2r^2 \int\limits_r^\infty \frac{\xi - 3m}{r^4}\,\mathrm{e}^F dr = \frac{2}{3}\frac{\xi - 3m}{r}\mathrm{e}^F + \frac{2r^2}{3}\int\limits_r^\infty \frac{\left[(\xi - 3m)\mathrm{e}^F\right]'}{r^3}\, dr$$

$$= \frac{2}{3}\frac{\xi - 3m}{r}\,\mathrm{e}^F + \frac{\mathrm{e}^{2F}}{3} + \frac{r^2}{3}\int\limits_r^\infty \left(\mathrm{e}^{2F}\right)' \frac{dr}{r^2} + \frac{2r^2}{3}\int\limits_r^\infty \frac{\xi - 3m}{r^3}\left(\mathrm{e}^F\right)' dr\,.$$

Because of nonnegativity of the integrals and Lemma 1, we have

$$A^2 > \frac{\xi - 3m}{r}\mathrm{e}^F\,. \tag{45}$$

The proof is complete. □

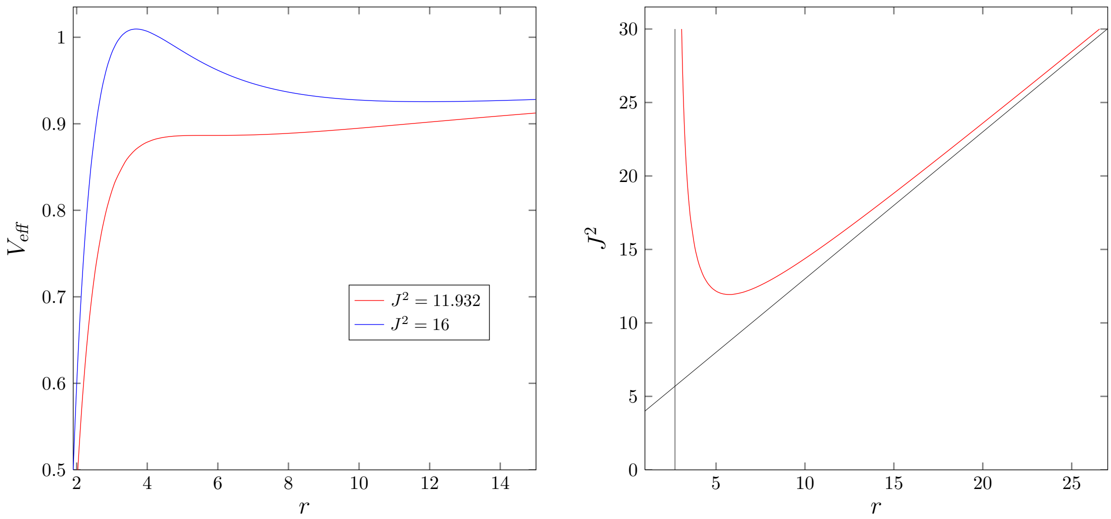

Figure 2: On the left: the effective potentials for the black hole with the Schwarzschild mass $m = 1$ and the scalar field ϕ_1 (see Figure 1) as the source. The minimum of the potential for $J^2 = 16$ corresponds to a stable circular orbit while the maximum corresponds to an unstable circular orbit; with decreasing J, these extrema approach each other and coincide for some $J_{ISCO} > 0$ at the inflection point ($J^2_{ISCO} = 11.932$, $r_{ISCO} = 5.741$) which determines the ISCO. On the right: the squared specific angular momentum J^2 as a function of the radius of a circular orbit; the minimum corresponds to the ISCO.

Now we can consider, with sufficient generality, the question of the existence and stability of circular orbits around black holes with classical scalar field hair. It is straightforward to check that

$$V'_{eff} = \left(A^2\right)' - \frac{2}{r}\frac{\xi - 3m}{r}\mathrm{e}^F \frac{J^2}{r^2} = \frac{2}{r}\left\{A^2 - \frac{\xi - 3m}{r}\mathrm{e}^F\left(1 + \frac{J^2}{r^2}\right)\right\}. \tag{46}$$

Since for any fixed J there exists $a = a(J) > 0$ such that $V'_{eff} > 0$ in the asymptotic region (a, ∞), and also $V'_{eff}(r_{ph}) > 0$, the effective potential either has no extrema or has at least one maximum and one minimum in the region $[r_{ph}, \infty)$. As a consequence of Lemma 2 and boundedness from above of the positive function $(\xi - 3m)\,\mathrm{e}^F/r^3$ in the region $[r_{ph}, \infty)$, there exists $J_{ISCO} > 0$ such that the extremums exist for all $J > J_{ISCO}$, but not for $J < J_{ISCO}$.

Let $\bar{r}$ be an extremum. Then the function $\bar{r}(J)$ is well-defined implicitly by the equation $V'_{eff} = 0$ (with V'_{eff} given by (46)) in some interval (J_{ISCO}, J_*) in which for a fixed mass we have

$$\frac{\partial \bar{r}}{\partial J} = -\left(\frac{\partial V'_{eff}}{\partial J} \Big/ \frac{\partial V'_{eff}}{\partial r}\right)_{r=\bar{r}} = \frac{4(\xi - 3m)\mathrm{e}^F J}{r^4 V''_{eff}}\Big|_{r=\bar{r}}.$$

Let $\bar{r}_1$ and $\bar{r}_2$ be, respectively, the leftmost maximum and the leftmost minimum of the effective potential. Thus, in the interval (J_{ISCO}, J_*),

$$\frac{\partial \bar{r}_1}{\partial J} < 0 \quad \text{and} \quad \frac{\partial \bar{r}_2}{\partial J} > 0\,,$$

so that the maximum and the minimum approach each other as $J \to J_{ISCO}$ from above and then coincide at the inflection point for $J = J_{ISCO} > 0$. This proves that the value $J = J_{ISCO}$ corresponds to the ISCO. Note that one can also consider the inverse function $J(\bar{r})$ defined in the interval (r_{ph}, ∞), and find that the parameters J_{ISCO} and r_{ISCO} are determined by the unique minimum of this function, as shown in Figure 2, because $V''_{eff} = 0$ and $\partial J/\partial\bar{r} = 0$ at the inflection point. Thus, Lemma 2 says, in essence, that a black hole supported by a classical scalar field has the ISCO only of the first kind.

Now we discuss briefly circular orbits around *naked singularities* and *regular solutions*. Generally, naked singularities have ISCOs of the second kind but some of them may not have the ISCO at all. We will say that a naked singularity is of general type if the metric function $A^2 = V_{eff}\big|_{J=0}$ has a minimum at some point $r_{ISCO} > 0$ which corresponds to the innermost orbit in the family of all possible stable circular orbits. Thus, there is the ISCO of the second kind: the specific angular momentum and the angular velocity of a test particle at the ISCO are zero, so that the particle will remain at rest for any static observer. Note that this feature of the ISCO are also possessed by some other types of naked singularities [24, 25, 26]. It is easy to see directly from (15) — (18) that a naked singularity is of general type if $\xi(0) > 3m$. If $\xi(0) = 3m$ then the corresponding naked singularity is of general type if and only if $\xi = \xi(0) + ar + br^2 + O\big(r^3\big)$ $(a > 0,\, b > 0)$ as $r \to 0$. If $\xi(0) = 3m$ and $\xi'(0) = 0$ then the corresponding naked singularity has stable circular orbits of any radius, that is, does not have any ISCO at all. At last, there are regular solutions, which necessarily obey the condition $\xi = \xi(0) + ar + br^4 + O\big(r^5\big)$ $(a > 0)$ as $r \to 0$; they also have stable circular orbits of any radius. In the last two cases, the specific angular momentum tends to zero as $r \to 0$.

5.2 Phantom scalar field configurations, $\varepsilon = -1$

Now we use the quasiglobal coordinates, so that the metric has the form (19) and any configuration is completely specified by its mass m and area function $C(r)$. In this case the problem is much more complicated because the behaviour of the metric function A^2 (and the corresponding effective potentials) is determined by the entire function $C(r)$ rather than its power series expansion up to some finite order. It means that now ISCOs cannot be characterized only in terms of a few parameters as it has been done in the previous section, where the type of ISCO was determined (for $\xi(0) \neq 3m$) by $\xi(0)$ and m. In fact, for any given admissible function $C(r)$, one has to examine the corresponding spacetime individually.

It follows from Eqs. (20) and the asymptotic behaviour (22) that $C'' \geqslant 0$ and $C' \leqslant r/C \leqslant 1$ in the region outside the event horizon of a black hole or outside the throat of a wormhole. Also,

$$\big(A^2\big)' = \frac{2}{C}\left(C'A^2 - \frac{r-3m}{C}\right),$$

and the necessary condition (35) for the existence of circular orbits now reads

$$AC' - A'C = \frac{1}{A}\left(A^2C' - \frac{1}{2}\left(A^2\right)'C\right) = \frac{r-3m}{AC} \geqslant 0\,,$$

so that we are only interested in the region $r \geqslant 3m$. Thus, in the quasiglobal coordinates, the radius of the photon circular orbit equals $3m$ as in the Schwarzschild spacetime. However, the ISCO properties and the behaviour of the circular orbits are very different from those for the Schwarzschild case. As we will see below, the condition $\left(A^2\right)' \geqslant 0$ does not, in general, hold in the region $r \geqslant 3m$ even for black holes. Since

$$V_{eff}' \;=\; \left(A^2\right)' - \frac{2}{C}\frac{r-3m}{C}\frac{J^2}{C^2} \;=\; \frac{2}{C}\left\{C'A^2 - \frac{r-3m}{C}\left(1+\frac{J^2}{C^2}\right)\right\}, \tag{47}$$

it means that for all $J \geqslant 0$ the effective potential has a minimum in the region if A^2 does. Thus, there can exist, depending on the mass m, both of the two above kinds of innermost stable circular orbits. Moreover, there exist configurations that do not have circular orbits at all. For example, the function $C(r) = \sqrt{r^2+a^2}$ determines the Ellis wormhole (named 'drainhole' by the author [27] and discovered independently by Bronnikov [12]) which is formed by the self-gravitating phantom massless scalar field. This solution has $A(r) = 1$, $V(\phi) = 0$, and $\phi(r) = \arctan(r/a) - \pi/2$. The corresponding spherically symmetric spacetime is curved, static, and geodesically complete but has no circular orbits.

In general, phantom scalar field configurations demonstrate a variety of possible behaviours in the asymptotic region $r \to -\infty$ [28, 29]. Also, such a configuration can either have or not have the event horizon. A regular asymptotically flat (near plus and minus infinity) wormhole may exist either if $m = 0$ and $C(r)$ is an even function, or if $C(r)$ has a minimum at some positive value of the radial coordinate; in the latter (fine-tuning) case the asymptotic regularity takes place only for a unique positive value of m observed in the half $r > 0$ of the wormhole, while in the 'negative' half the mass has a negative value.

We restrict our attention to phantom scalar field black holes. It is convenient to illustrate the basic properties of the ISCOs by a simple example which can be treated analytically. We choose

$$C \;=\; \left(r^4 + 2r^2 + 16\right)^{1/4} \tag{48}$$

and find from (20) that

$$\phi\,' \;=\; -\,\frac{\sqrt{r^4+47r^2+16}}{r^4+2r^2+16}\,, \tag{49}$$

$$A^2 = \frac{\sqrt{r^4+2r^2+16}}{240}\left[\left(18\sqrt{10}\,m+16\sqrt{15}\right)\arctan\left(\frac{r\sqrt{10}+\sqrt{15}}{5}\right)\right.$$
$$+\left(18\sqrt{10}\,m-16\sqrt{15}\right)\arctan\left(\frac{r\sqrt{10}-\sqrt{15}}{5}\right)$$
$$\left.+\,15\sqrt{6}\,m\ln\frac{r^2+r\sqrt{6}+4}{r^2-r\sqrt{6}+4}-18\sqrt{10}\,\pi m\right]. \tag{50}$$

For $m = 0$ all these functions are even and the spacetime is a symmetric wormhole or the corresponding topological geon — the quotient spacetime by the isometry $r \to -r$. There are no stable circular orbits, but for any $J \geqslant 0$ there exists an unstable circular orbit placed at $r = 0$, that is, in the throat of the wormhole or on the threshold hypersurface of the geon where V_{eff} has a unique maximum. For $m > 0$, a number of results related to the innermost circular orbits are illustrated in Figures 3–7.

In the interval of masses $0 < m \leqslant m_c$, $m_c \approx 0.2254$, there are ISCOs of the second kind (see Figures 4, 5 and 6). For $m \geqslant m_c$ the corresponding black holes have ISCOs only of the first kind, as can be seen from Figures 7. In any case, the curve $J^2(r)$ versus orbital radius r has a unique minimum which may be negative, zero, or positive; the latter corresponds to black holes with the ISCO of the first kind. Negative values of J^2 are unphysical. Also, it follows from the expression (47) and the asymptotic relation (22) that in any case $J^2 = mr + 3m^2 + o(1)$ as $r \to +\infty$, and $J^2 \to +\infty$ as $r \to 3m$ from the right. The function $r_{ISCO}(m)$ has a unique minimum at $m = m_c$ and goes to infinity as $m \to +0$ or $m \to +\infty$, as shown in Figure 8.

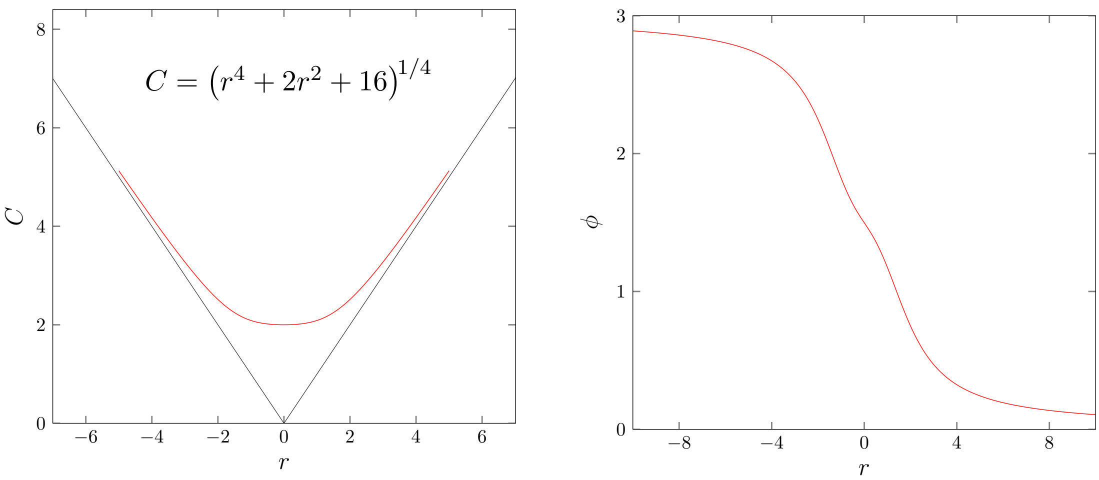

Figure 3: The wormhole (or the corresponding topological geon) is determined by the even function $C(r)$. The field ϕ, obtained by integrating (49), obeys the asymptotic conditions $\phi \to 0$ as $r \to +\infty$ and $\phi \to 2.996$ as $r \to -\infty$.

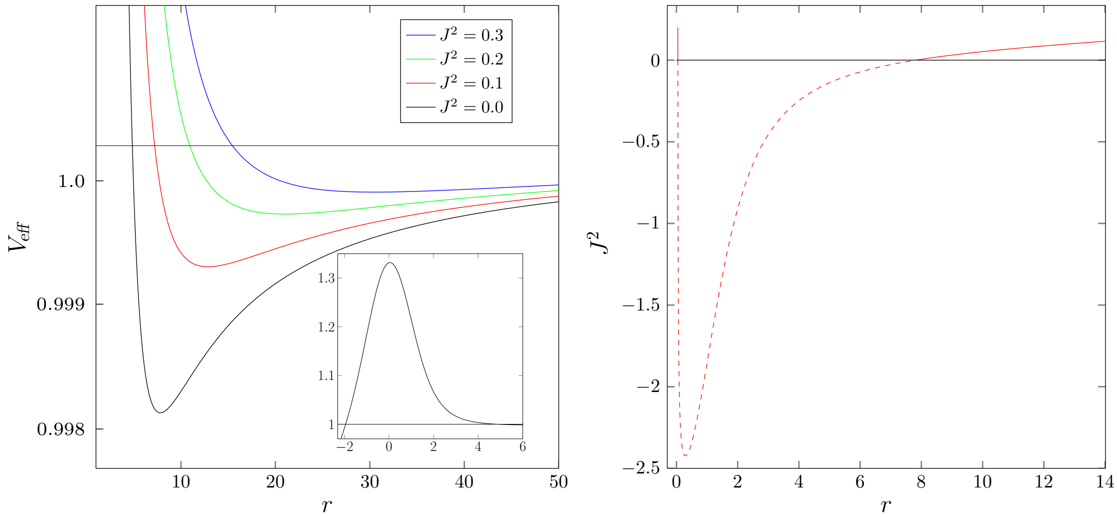

Figure 4: The black hole (48) – (50) with $m = 0.01$. The corresponding ISCO has $J_{ISCO} = 0$, $r_{ISCO} = 7.820$, that is, it is of the second kind. Stable circular orbits exist for any $J > 0$. For $0.0450 > r > 0.03 = 3m$ there also exist unstable circular orbits for which $J \to +\infty$ as $r \to 3m + 0$. The inset shows the typical global behaviour (for $J = 0$, as an example) of the potential which reaches a maximum and then goes to zero as r goes to the horizon ($r_h = -8.136$) from the right.

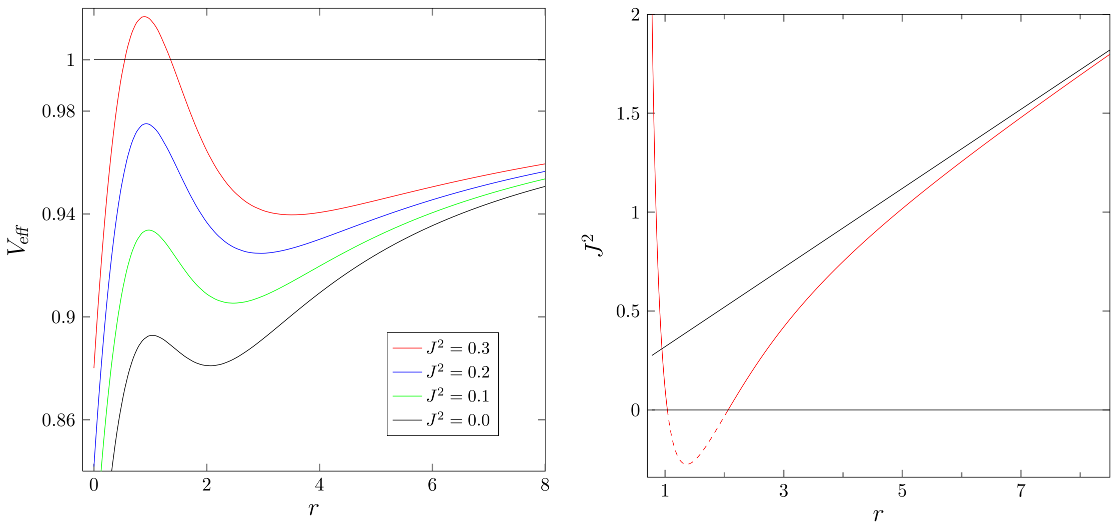

Figure 5: The black hole (48) – (50) with $m = 0.2$ is qualitatively similar to the previous one with $m = 0.01$. Now $J_{ISCO} = 0$, $r_{ISCO} = 2.062$, $r_h = -1.364$, and unstable circular orbits exist for $1.038 > r > 0.6 = 3m$.

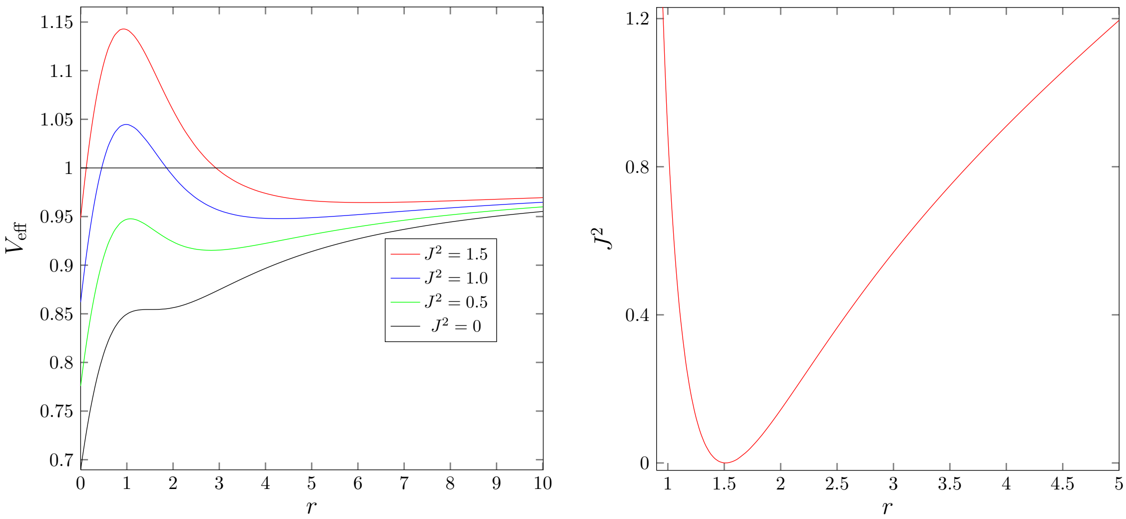

Figure 6: The black hole (48) – (50) with the transition mass $m = m_c = 0.2254$ has the ISCO parameters $J_{ISCO} = 0$ and $r_{ISCO} = r_c = 1.512$; the latter is an inflection point of the effective potential. In the intervals of masses $m > m_c$ and $0 < m \leqslant m_c$, the corresponding ISCOs are, respectively, of the first and second kind.

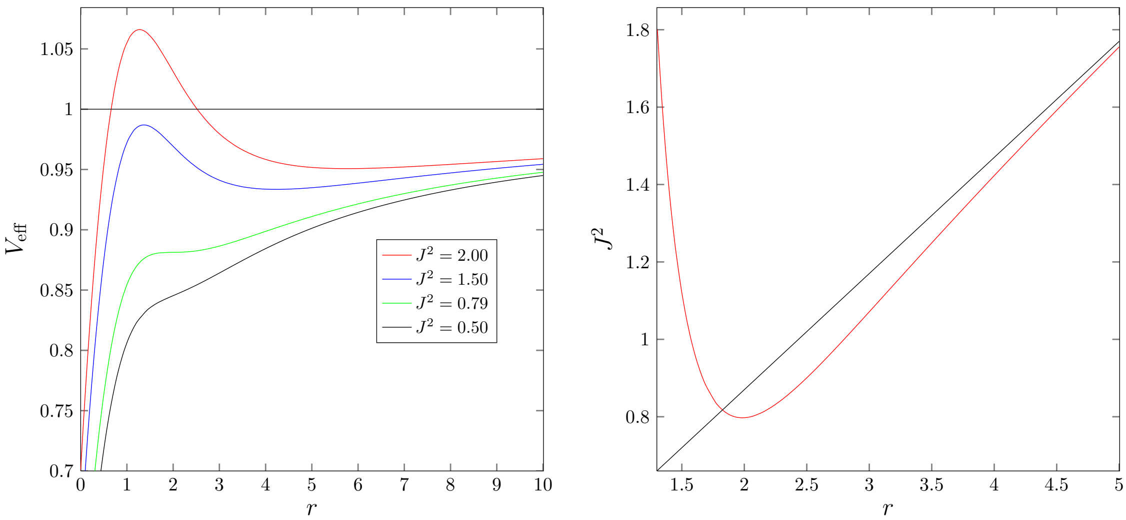

Figure 7: The black hole (48) – (50) with $m = 0.3 > m_c$. The corresponding ISCO (of the first kind) has the parameters $J_{ISCO} = 0.7973$ and $r_{ISCO} = 1.984$. Stable circular orbits exist for any $J > J_{ISCO}$.

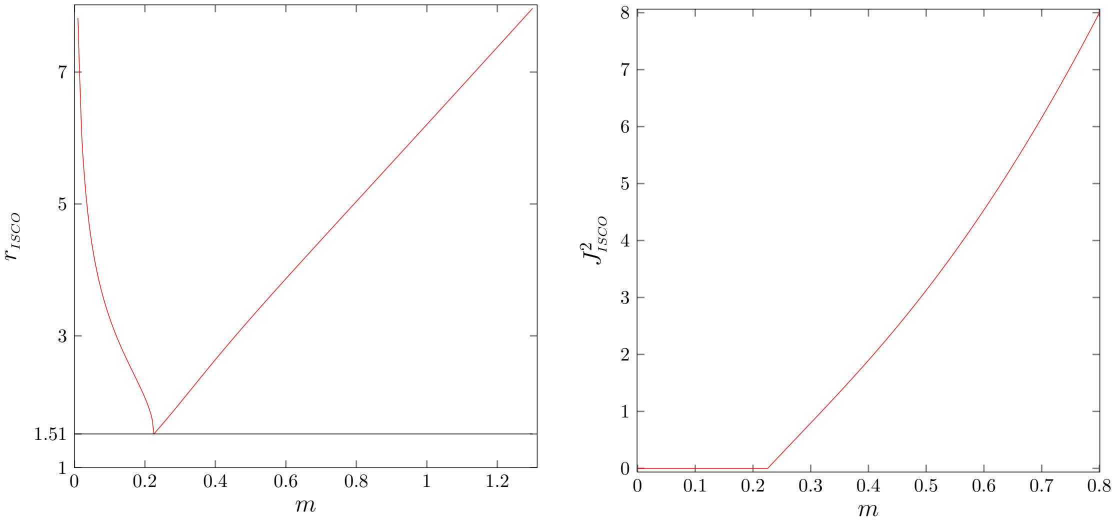

Figure 8: On the left: the function $r_{ISCO}(m)$ has a unique minimum $r_c = 1.512$ at $m = m_c = 0.2254$ and goes to infinity as m goes to zero or infinity. On the right: $J^2_{ISCO}(m)$ goes to infinity as m goes to infinity. Both curves have a cusp at $m = m_c$.

6. Concluding remarks

In this article we have presented a general method for studying circular orbits, particularly ISCOs, around static, spherically symmetric scalar field configurations, both for classical and phantom scalar fields minimally coupled to gravity. A key feature of the method is its independence from the form of the self-interaction potential. In the direct problem for the Einstein-Klein-Gordon equations, one has previously to define the self-interaction potential, to impose asymptotic conditions, and also then to find solutions. Instead we use the quadrature formulas for the metric functions, which gives us rich possibilities to model scalar hairy configurations in terms of a very small number of parameters or the area metric function (which has a very simple behaviour), in the cases of classical and phantom scalar fields respectively. This approach in turn have enabled us to study circular orbits around scalar hairy configurations in general, that is, in some sense for all admissible potentials at the same time. In particular, we have found that there exist two kinds of ISCOs, with nonzero and zero specific angular momentum of a test particle, referred to as the first and second kinds respectively.

Classical scalar field black holes have ISCOs only of the first kind. It is well-known that the Schwarzschild naked singularities have negative masses and have no circular orbits at all. On the contrary, classical scalar field naked singularities of general type have positive masses, circular orbits, and ISCOs of the second kind; some fine-tuning naked singularities and regular configurations have stable circular orbits of any positive radius, that is, do not have ISCOs. Black holes supported by phantom scalar fields can have ISCOs of both the kind depending on the Schwarzschild mass. In a family of black holes with one and the same area function, the ISCOs are of the second kind for small values of the mass, while they are of the first kind for masses greater than some intermediate value.

We hope that systematic investigations of scalar hairy object and circular orbits around them are physically motivated and will be practically useful in astronomical observations in the near future. In this connection it would also be interesting to study a number of important mathematical questions which have not been considered in the article: for example, whether the effective potential of a black hole with classical scalar hair can have more than two extremums. In other words, there remains the question of the existence of two different stable circular orbits with one and the same specific angular momentum of a test particle.

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