http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 3, No 3, p. 19 – 28 (2015)

On the optimal drawings of Cartesian products of special 6-vertex graphs with path

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Received 2 December 2015, in final form 23 December. Published 24 December 2015.

Abstract. Finding minimum necessary intersections in graph representations is useful in many areas. The most prominent areas are automated graph drawings and VLSI-layouts. The exact value of the crossing number is known only for few classes of graphs, mainly with regular structure such as various products of graphs. Among the products of graphs, the Cartesian product has received great attention in the mathematical publications. Klešč, Jendroľ and Ščerbová determined the crossing numbers of Cartesian products of paths with all graphs of order at most four and with all connected graphs on five vertices. Moreover, the crossing numbers of Cartesian products of paths with some graphs of order six are known. In the paper, we extend these results by determining crossing numbers of Cartesian products $G \Box P_n$ for several other graphs G on six vertices.

Keywords: graph, drawing, crossing number

MSC numbers: 05C10, 05C38

The research was supported by the Slovak VEGA grant No. 1/0389/15.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. A *drawing* of the graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. In such a drawing, the intersection of the interiors of the arcs is called a *crossing*. We assume that in a drawing no edge passes through any vertex other than its end-points, no two edges touch each other (i.e., if two edges have a common interior point, then they cross properly at this point), and no three edges cross at the same point. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross each other. The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a good drawing of G in the plane. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $\mathrm{cr}_D(G_i)$ the number of crossings among edges of G_i in D. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The Cartesian product $G_1 \Box G_2$ of the graphs G_1 and G_2 has vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are adjacent in $G_1 \Box G_2$ if and only if either $u = v$ and u' is adjacent with v' in G_2, or $u' = v'$ and u is adjacent with v in G_1. Let C_n be the cycle of length n, P_n be the path of length n, and S_n be the star isomorphic to $K_{1,n}$. In the proofs of the paper, we will often use the term “region” also in nonplanar drawings. In this case, crossings are considered to be vertices of the “map”.

The crossing number of a graph is an important property, which can be employed in computer science in many areas. The most prominent areas are automated graph drawings and VLSI-layouts. It is well known that the problem of determination of the crossing numbers of graphs is NP-complete (see [3]) and it remains NP–hard even for cubic graphs [4]. The lower bound on the chip area is determined by crossing number and by number of vertices of the graph [1, 11]. It plays an important role in various fields of discrete/computational geometry [12]. The crossing number is also a parameter yielding the deviation of the graph from being planar. The crossing number significantly influences readability and therefore it is the most important parameter when considering aesthetics of a graph. It is mostly used in automated visualisation of graphs.

In the next sections, we give the crossing numbers of Cartesian products of paths with four graphs on six vertices shown in Fig. 1.

2. Preliminary results

In this section, we will prove some lemmas, which help us to give the crossing numbers of Cartesian products of paths with four graphs G, H, J_1 and J_2 on six

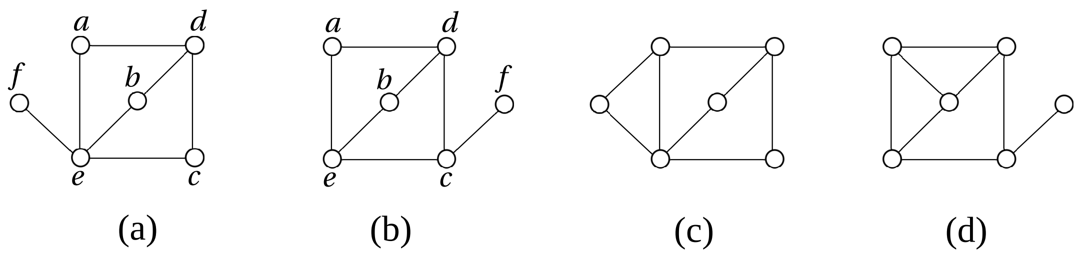


Figure 1: The graphs G, H, J_1 and J_2 on six vertices.

vertices shown in Fig. 1. We assume $n \geq 1$ and find it convenient to consider the graph $G \square P_n$ in the following way: it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies G^i, $i = 0, 1, \ldots, n$, of graph G and in six paths of length n, see Fig. 2(a). For $i = 0, 1, \ldots, n$, let a_i, b_i and c_i be the vertices of G^i of degree two, d_i the vertex of degree three, e_i the vertex of degree four, and f_i the vertex of degree one (see Fig. 1(a)). Let us denote by M_G^i the subgraph of $G \square P_n$ containing the vertices of G^{i-1} and G^i and six edges joining G^{i-1} to G^i, $i = 1, 2, \ldots, n$. Let Q_G^i, $i = 1, 2, \ldots, n-1$, denote the subgraph of $G \square P_n$ induced by $V(G^{i-1}) \cup V(G^i) \cup V(G^{i+1})$. So, $Q_G^i = G^{i-1} \cup M_G^i \cup G^i \cup M_G^{i+1} \cup G^{i+1}$.

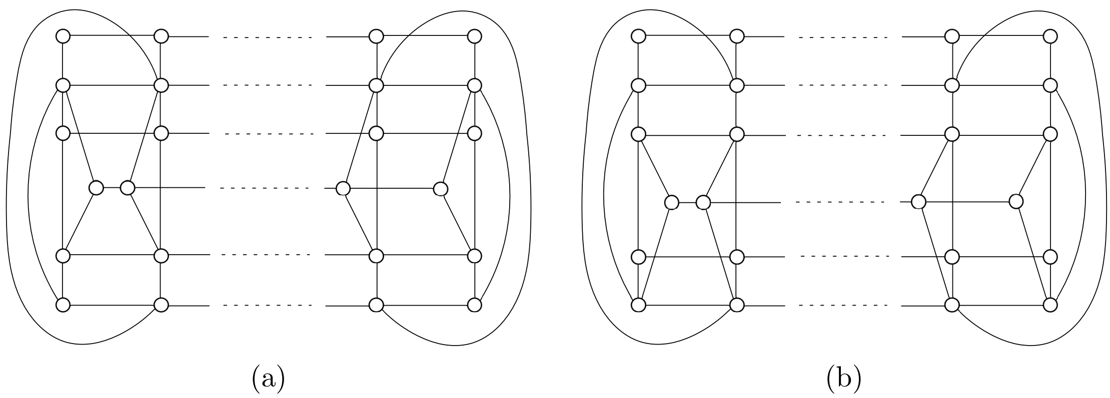


Figure 2: The graphs $G \square P_n$ and $H \square P_n$.

Similarly, the graph $H \square P_n$ has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies H^i of graph H and six paths of length n, see Fig. 2(b). For $i = 0, 1, \ldots, n$, let a_i and b_i be the vertices of H^i of degree two, c_i, d_i and e_i the vertex of degree three, and f_i the vertex of degree one (see Fig. 1(b)). For $i = 1, 2, \ldots, n$, let M_H^i denote the subgraph of $H \square P_n$ consisting of the vertices in H^{i-1} and H^i and of the edges joining H^{i-1} with H^i, and let $Q_H^i = H^{i-1} \cup M_H^i \cup H^i \cup M_H^{i+1} \cup H^{i+1}$.

Both graphs $G \square P_n$ and $H \square P_n$ contain $K_{2,3} \square P_n$ as a subgraph. For $i = 0, 1, \ldots, n$, let $K_{2,3}^i$ denote the complete bipartite subgraph of the graph $K_{2,3} \square P_n$ and let M_K^i denote the corresponding subgraph of M_G^i or M_H^i. For $i = 0, 1, \ldots, n$, let a_i, b_i

and c_i be the vertices of $K^i_{2,3}$ of degree two, and d_i and e_i the vertices of degree three. In a good drawing D, we say that a graph $K^i_{2,3}$ *separates* the graphs $K^k_{2,3}$ and $K^l_{2,3}$ (the vertices of a graph $K^k_{2,3}$) if there exists a cycle C of $K^i_{2,3}$ such that $K^k_{2,3}$ and $K^l_{2,3}$ (the vertices of a graph $K^k_{2,3}$) are contained in different components of $\mathbb{R}^2 \setminus C$. Consider the graph $K_{2,3}\Box P_2$ which is a subgraph of Q^i_G in $G\Box P_n$ as well as a subgraph of Q^i_H in $H\Box P_n$, $i = 1, 2, \dots, n-1$. The following results enable us to simplify the proofs in the next sections.

Lemma 1. *Let D be a good drawing of the graph $K_{2,3}\Box P_n$, $n \geq 2$, in which each of the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$, $i = 1, 2, \dots, n-1$, has at most two crossings on its edges. Then $K^{i-1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i+1}_{2,3}$, $K^{i+1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i-1}_{2,3}$, and if $K^i_{2,3}$ has an internal crossing, $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$.*

Proof. If $K^{i-1}_{2,3}$ separates $K^i_{2,3}$ and $K^{i+1}_{2,3}$ (if $K^{i+1}_{2,3}$ separates $K^i_{2,3}$ and $K^{i-1}_{2,3}$), then the graph $K^{i-1}_{2,3}$ ($K^{i+1}_{2,3}$) is crossed by all five edges joining the separated graphs. This contradicts the assumption that every graph $K^i_{2,3}$ has at most two crossings on its edges. It remains to show that $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$ if $K^i_{2,3}$ has an internal crossing. Without loss of generality, let $K^{i-1}_{2,3}$ is placed inside $K^i_{2,3}$. The subdrawing of $K^i_{2,3}$ induced by D divides the plane in such a way that on the boundary of every region inside $K^i_{2,3}$ there are at most three vertices of $K^i_{2,3}$. Thus, one of graphs $K^{i-1}_{2,3}$ and M^i_K crosses $K^i_{2,3}$ at least twice. So, the number of crossings on the edges of $K^i_{2,3}$ is at least three, a contradiction. □

Lemma 2. *Let D be a good drawing of the graph $K_{2,3}\Box P_n$, $n \geq 2$, in which each of the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$, $i = 1, 2, \dots, n-1$, has at most two crossings on its edges and none of them separates two other. Then $K^{i-1}_{2,3}$, $K^i_{2,3}$ and $K^{i+1}_{2,3}$ do not cross each other.*

Proof. Assume a good drawing D of the graph $K_{2,3}\Box P_n$, $n \geq 2$, in which each of the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$, $i = 1, 2, \dots, n-1$, has at most two crossings on its edges and none of them separates two other. If two of the 2-connected graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$ cross, then they cross at least twice. So, none of $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$ crosses both others. Moreover, if two graphs of $K^{i-1}_{2,3}$, $K^i_{2,3}$, and $K^{i+1}_{2,3}$ cross, then none of them has an internal crossing. If $\mathrm{cr}_D(K^i_{2,3}, K^{i-1}_{2,3}) \neq 0$, then the subdrawing of $K^i_{2,3}$ induced by D divides the plane as shown in Fig. 3(a) and, as $\mathrm{cr}_D(K^i_{2,3}, K^{i+1}_{2,3}) = 0$, in D at least one edge of M^{i+1}_K joining $K^i_{2,3}$ to $K^{i+1}_{2,3}$ crosses $K^i_{2,3}$. This contradicts the assumption of at most two crossings on the edges of $K^i_{2,3}$. The same contradiction is obtained if $\mathrm{cr}_D(K^i_{2,3}, K^{i+1}_{2,3}) \neq 0$. The last possibility is that $\mathrm{cr}_D(K^{i-1}_{2,3}, K^{i-1}_{2,3}) \neq 0$. In this case $K^{i-1}_{2,3}$ divides the plane as shown in Fig. 3(a) and at least one edge of M^i_K crosses $K^{i-1}_{2,3}$. This contradiction completes the proof. □

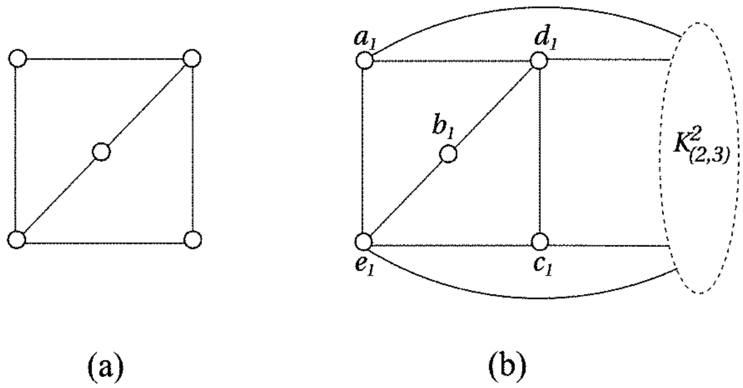


Figure 3: The unique planar drawing of $K_{2,3}$ and the graph $K^1_{2,3} \cup M^2_K \cup K^2_{2,3}$ without the edge b_1b_2.

Lemma 3. *Let D be a good drawing of the graph $K_{2,3}\Box P_2$ in which every subgraph $K^i_{2,3}$, $i = 0, 1, 2$, has at most two crossings on its edges. If the subgraphs $K^0_{2,3}$, $K^1_{2,3}$, and $K^2_{2,3}$ do not cross each other and none of them separates two other, then $\mathrm{cr}_D(K^1_{2,3}) + \mathrm{cr}_D(K^1_{2,3}, M^1_K \cup M^2_K) + \mathrm{cr}_D(K^0_{2,3} \cup M^1_K, K^2_{2,3} \cup M^2_K) \geq 3$.*

Proof. Assume that there is a good drawing D of the graph $K_{2,3}\Box P_2$ in which two different complete bipartite graphs do not cross each other and none of the graphs $K^0_{2,3}$, $K^1_{2,3}$ and $K^2_{2,3}$ separates two other and that $\mathrm{cr}_D(K^1_{2,3}) + \mathrm{cr}_D(K^1_{2,3}, M^1_K \cup M^2_K) + \mathrm{cr}_D(K^0_{2,3} \cup M^1_K, K^2_{2,3} \cup M^2_K) \leq 2$. As $K_{2,3}$ is not outerplanar graph, either $K^1_{2,3}$ has an internal crossing or $K^1_{2,3}$ crosses $M^1_K \cup M^2_K$ at least twice. In the first case, at least one of $K^0_{2,3} \cup M^1_K$ and $M^2_K \cup K^2_{2,3}$ does not cross $K^1_{2,3}$. Without loss of generality, let $\mathrm{cr}_D(K^1_{2,3}, K^0_{2,3} \cup M^1_K) = 0$. Then the subdrawing of $K^0_{2,3} \cup M^1_K \cup K^1_{2,3}$ induced by D divides the plane in such a way that on the boundary of every region outside $K^1_{2,3}$ there are at most two vertices of $K^1_{2,3}$. The complete bipartite graph $K^2_{2,3}$ does not cross an edge of the 2-connected subgraph $K^0_{2,3} \cup M^1_K \cup K^1_{2,3}$, otherwise $\mathrm{cr}_D(K^2_{2,3}, K^0_{2,3} \cup M^1_K \cup K^1_{2,3}) \geq 2$ and $\mathrm{cr}_D(K^1_{2,3}) + \mathrm{cr}_D(K^1_{2,3}, M^1_K \cup M^2_K) + \mathrm{cr}_D(K^0_{2,3} \cup M^1_K, K^2_{2,3} \cup M^2_K) \geq 3$, a contradiction. Thus, $K^2_{2,3}$ is placed in one region outside $K^1_{2,3}$. But, in this case, at least two edges of M^2_K joining $K^2_{2,3}$ with the vertices of $K^1_{2,3}$ cross the edges of $K^0_{2,3} \cup M^1_K \cup K^1_{2,3}$ and $\mathrm{cr}_D(K^1_{2,3}) + \mathrm{cr}_D(K^1_{2,3}, M^1_K \cup M^2_K) + \mathrm{cr}_D(K^0_{2,3} \cup M^1_K, K^2_{2,3} \cup M^2_K) \geq 3$. This forced that $\mathrm{cr}_D(K^1_{2,3}) = 0$ and that $K^1_{2,3}$ crosses $M^1_K \cup M^2_K$ at least twice. As $K^1_{2,3}$ does not have an internal crossing, its unique planar drawing is shown in Fig. 3(a). Without loss of generality, let the edge b_1b_2 of M^2_K crosses $K^1_{2,3}$. The unique subdrawing of $(K^1_{2,3} \cup M^2_K \cup K^2_{2,3}) \smallsetminus \{b_1b_2\}$ in Fig. 3(b) shows that on the boundary of every region outside $K^1_{2,3}$ there are at most two vertices of $K^1_{2,3}$. Thus, as only one edge of M^1_K can cross $K^1_{2,3}$, it is easy to see that $\mathrm{cr}_D(K^0_{2,3} \cup M^1_K, K^2_{2,3} \cup M^2_K) \neq 0$ and the proof is done. □

3. The crossing number of $G\square P_n$

The crossing number of the graph $G\square P_1$ is two, because it contains $K_{2,3}\square P_1$ as a subgraph and $\mathrm{cr}(K_{2,3}\square P_1) = 2$ (see [8]). The reverse inequality $\mathrm{cr}(G\square P_1) \leq 2$ implies from a suitable drawing of the graph $G\square P_1$ with two crossings. In Fig. 2(a) there is the drawing of the graph $G\square P_n$ with $3n-1$ crossings. The next result is fundamental in proving that the crossing number of the graph $G\square P_n$ is $3n-1$ for $n \geq 2$.

Lemma 4. *If D is a good drawing of the graph $G\square P_n$, $n \geq 2$, in which each of the subgraphs G^i, $i = 0, 1, 2, ..., n$, has at most two crossings on its edges, then in D there are at least $3n-1$ crossings.*

Proof. In a drawing of the graph $G\square P_n$, let us consider the following types of possible crossings on the edges of Q^i_G for all $i = 1, 2, ..., n-1$:

(1) a crossing of an edge in $G^{i-1} \cup M^i_G$ with an edge in $G^{i+1} \cup M^{i+1}_G$,

(2) a crossing of an edge in $M^i_G \cup M^{i+1}_G$ with an edge in G^i,

(3) a crossing among the edges of G^i.

It is readily seen that each of the considered crossings appears in a good drawing of the graph $G\square P_n$ only on the edges of one subgraph Q^i_G. Consider now a good drawing D of $G\square P_n$ satisfying the assumptions in Lemma 4. By Lemma 1, in the subdrawing $D(Q^i_G)$ of the subgraph Q^i_G induced by D, $i = 1, 2, \ldots, n-1$, $K^{i-1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i+1}_{2,3}$, $K^{i+1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i-1}_{2,3}$, and if $K^i_{2,3}$ has an internal crossing, $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$. It remains to prove that $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$ if $K^i_{2,3}$ has the planar drawing. Let $K^i_{2,3}$ separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$, and $K^i_{2,3}$ has the planar drawing. Then a path $e_{i-1}f_{i-1}f_if_{i+1}e_{i+1}$ crosses the edges of $K^i_{2,3}$ at least once. Hence, the assumption of at most two crossings on $K^i_{2,3}$ forced that none of $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$ crossed $K^i_{2,3}$. The unique planar drawing of $K^i_{2,3}$ in Fig. 3(a) divides the plane in such a way that on the boundary of every region $K^i_{2,3}$ there are at most four vertices of $K^i_{2,3}$. If the graphs $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$ are placed in different regions of $K^i_{2,3}$, the graph $M^i_K \cup M^{i+1}_K$ crosses $K^i_{2,3}$ at least twice. Thus, the graph $K^i_{2,3}$ has at least three crossings on its edges and this contradicts the assumption that every subgraph G^i has at most two crossings on its edges. Thus, none of the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$ and $K^{i+1}_{2,3}$ separates two other and by Lemma 2, they do not cross each other.

So, by Lemma 3, every subdrawing $D(Q^i_G)$, $i = 1, 2, \ldots, n-1$, contains at least three crossings, every of types (1), (2) or (3). This enforces that, there are at least $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q^i_G) = 3(n-1)$ crossings among the edges of the subgraph $G^0 \cup M^1_G \cup G^1 \cup \cdots \cup G^{n-1} \cup M^n_G \cup G^n$, in D. It remains to prove that, in D, every of the subgraphs Q_G^1 and Q_G^{n-1} has at least one crossings which is not counted in $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q_G^i)$. If it is true, then in D there are at least $3n-1$ crossings and we are done.

Consider the subgraph Q_G^1. We have shown above, that the complete bipartite graphs $K_{2,3}^0$, $K_{2,3}^1$ and $K_{2,3}^2$ do not cross each other and none of them separates two other. So, by Lemma 3, the subdrawing $D(Q_G^1)$ contains at least three crossings of types (1), (2) or (3). The aim of this part of the proof is to show that there is at least one crossing not counted in $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q_G^i)$. If the subgraph $K_{2,3}^0$ has an internal crossing, we are done. Otherwise its unique planar drawing in Fig. 3(a) shows that there is at least one crossing between the edges of $K_{2,3}^0$ and the edges of M_G^1. This crossing is not counted in $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q_G^i)$. The same holds for the subgraph Q_G^{n-1} and so, in D there are at least $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q_G^i) + 2 = 3(n-1) + 2 = 3n-1$ crossings. This completes the proof.

□

Now we are able to give tha exact value of the crossing number for the graph $G \square P_n$.

Theorem 1. $\mathrm{cr}(G \square P_n) = 3n - 1$ *for* $n \geq 1$.

Proof. The drawing in Fig. 2(a) shows that $\mathrm{cr}(G \square P_n) \leq 3n - 1$, because every copy of G^i, $i = 1, 2, 3, ..., n-1$, is crossed three times, G^0 and G^n are crossed once and there is no other crossing in the drawing. We prove the reverse inequality by induction on n. The graph $K_{2,3} \square P_1$ is a subgraph of $G \square P_1$ and we know that $\mathrm{cr}(K_{2,3} \square P_1) = 2$ (see [8]). Thus, the crossing number of $G \square P_1$ is at least two and the result is true for $n = 1$. Assume that it is true for $n = k$, $k \geq 2$, and suppose there is a good drawing of the graph $G \square P_{k+1}$ with fewer than $3k + 2$ crossings. By Lemma 4, some of the subgraphs G^i, $i = 0, 1, ..., k+1$, must be crossed at least three times. By the removal of all edges of this G^i, we obtain a graph which is homeomorphic to $G \square P_k$ or a graph containing the subgraph $G \square P_k$. Both have fewer than $3k + 2$ crossings. This contradiction with the induction hypothesis completes the proof. □

4. The crossing number of $H \square P_n$

The crossing number of the graph $H \square P_1$ is at least two, because it contains the graph $K_{2,3} \square P_1$ as a subgraph and $\mathrm{cr}(K_{2,3} \square P_1) = 2$ (see [8]). The reverse inequality $\mathrm{cr}(H \square P_1) \leq 2$ follows from a suitable drawing of the graph $H \square P_1$ with two crossings. In Fig. 2(b) there is the drawing of the graph $H \square P_n$ with $3n - 1$ crossings. The next result is fundamental in proving that the crossing number of the graph $H \square P_n$ is $3n - 1$ for $n \geq 2$.

Lemma 5. *If D is a good drawing of the graph $H\square P_n$, $n \geq 2$, in which each of the subgraphs H^i, $i = 0, 1, 2, ..., n$, has at most two crossings on its edges, then in D there are at least $3n - 1$ crossings.*

Proof. In a drawing of the graph $H\square P_n$, let us consider the following types of possible crossings on the edges of Q^i_H, $i = 1, 2, ..., n - 1$:

(1) a crossing of an edge in $H^{i-1} \cup M^i_H$ with an edge in $H^{i+1} \cup M^{i+1}_H$,

(2) a crossing of an edge in $M^i_H \cup M^{i+1}_H$ with an edge in H^i,

(3) a crossing among the edges of H^i.

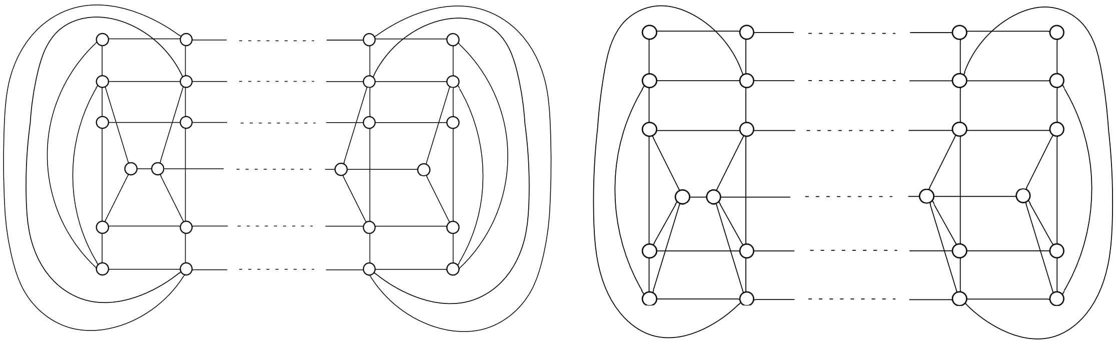
It is readily seen that every crossing of types (1), (2), and (3) appears in a good drawing of the graph $H\square P_n$ only on the edges of one subgraph Q^i_H. Consider now a good drawing D of $H\square P_n$ assumed in Lemma 5. Similarly, like in Lemma 4, the aim of this part of the proof is to show that the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$ and $K^{i+1}_{2,3}$ do not cross each other and none of them separates two other. By Lemma 1, in the subdrawing $D(Q^i_H)$ of the subgraph Q^i_H induced by D, $i = 1, 2, \ldots, n - 1$, $K^{i-1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i+1}_{2,3}$, $K^{i+1}_{2,3}$ does not separate $K^i_{2,3}$ and $K^{i-1}_{2,3}$, and if $K^i_{2,3}$ has an internal crossing, $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$. It remains to prove, that $K^i_{2,3}$ does not separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$ if $K^i_{2,3}$ has the planar drawing. Let $K^i_{2,3}$ separate $K^{i-1}_{2,3}$ and $K^{i+1}_{2,3}$, and $K^i_{2,3}$ has the planar drawing. Then a path $c_{i-1}f_{i-1}f_if_{i+1}c_{i+1}$ crosses the edges of $K^i_{2,3}$ at least once. Similarly, like in Lemma 4, we can show that the graph $K^i_{2,3}$ must have at least three crossings on its edges and this contradicts the assumption that every subgraph H^i has at most two crossings on its edges. It implies that none of the complete bipartite graphs $K^{i-1}_{2,3}$, $K^i_{2,3}$ and $K^{i+1}_{2,3}$ separates two other and by Lemma 2, they do not cross each other.

So, by Lemma 3, every subdrawing $D(Q^i_H)$, $i = 1, 2, \ldots, n - 1$, contains at least three crossings, every of types (1), (2) or (3). This forces that, in D, there are at least $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q^i_H) = 3(n - 1)$ crossings among the edges of the subgraph $H^0 \cup M^1_H \cup H^1 \cup \cdots \cup H^{n-1} \cup M^n_H \cup H^n$. It remains to prove that, in D, every of the subgraphs Q^1_H and Q^{n-1}_H has at least one crossing which is not counted in $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q^i_H)$. To show this, we can use the same consideration as in the proof of Lemma 4. So, in D there are at least $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q^i_H) + 2 = 3(n - 1) + 2 = 3n - 1$ crossings. This completes the proof.

□

Using Lemma 5, in a similar way as in Theorem 1 we can obtain the next results.

Theorem 2. $\mathrm{cr}(H\square P_n) = 3n - 1$ *for* $n \geq 1$.



(a) (b)

Figure 4: The graphs $J_1 \square P_n$ and $J_2 \square P_n$.

5. The crossing number of $J_1 \square P_n$ and $J_2 \square P_n$

The crossing number of $G \square P_n$ and $H \square P_n$ are given in the previous sections. These results enable us to determine the crossing numbers for Cartesian products of paths with two other graphs J_1 and J_2 of order six, see Fig. 1(c),(d).

The graph J_1 contains the graph G presented in Fig. 1(a) as a subgraph and therefore, $G \square P_n \subset J_1 \square P_n$. It was shown in Section 3 that $\mathrm{cr}(G \square P_n) = 3n - 1$. So, the crossing number of $J_1 \square P_n$ is at least $3n - 1$. The reverse inequality follows from the drawing in Fig. 4(a).

Similarly, the graph J_2 contains the graph H presented in Fig. 1(b) as a subgraph and therefore, $H \square P_n \subset J_2 \square P_n$. As $\mathrm{cr}(H \square P_n) = 3n - 1$ (see Theorem 2), the crossing number of $J_2 \square P_n$ is at least $3n - 1$. The reverse inequality follows from the drawing in Fig. 4(b).

Acknowledgements. The author is grateful to RNDr. Daniela Kravecová, PhD. for reading the paper and useful comments.

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