http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 3, No 3, p. 1 – 18 (2015)

Combinatorial and operator approaches to RED modeling

A. V. Korolkova[1,a], **D. S. Kulyabov**[1,2,b] **and L. A. Sevastianov**[1,3,c]

[1] Department of Applied Probability and Informatics, Peoples' Friendship University of Russia, Miklukho-Maklaya str. 6, Moscow, 117198, Russia

[2] Laboratory of Information Technologies, Joint Institute for Nuclear Research, Joliot-Curie 6, Dubna, Moscow region, 141980, Russia

[3] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Joliot-Curie 6, Dubna, Moscow region, 141980, Russia

e-mail: [a] akorolkova@sci.pfu.edu.ru, [b] yamadharma@gmail.com, [c] leonid.sevast@gmail.com

Received 1 December 2015, in final form 21 December. Published 23 December 2015.

Abstract. In the study of a phenomenon it is necessary to construct a model of this phenomenon. The models differ in their expressive properties, degree of maturity, requirements. We need quantitative and qualitative studies of models. We want to get the whole range of models from first principles. One of these methodologies is the method of stochastization of one-step processes. As a result of application of the methodology to the investigated phenomena we can get more than one mathematical model (the whole family of models). We can specify the preferred methods of investigation for each model of the family. Thus, we are expanding the range of methods for studying the phenomenon under investigation. In this article, using RED algorithm as an example, we will illustrate different approaches to framework of the method of stochastization of one-step processes.

Keywords: stochastic differential equations, master equation, Fokker–Planck equation, active queue management, random early detection
PACS numbers: 02.50.Ey, 02.30.Yy, 12.38.Bx

The authors are partially supported by the Russian Foundation for Basic Research (grant 15-07-08795).The second and third authors are partially supported by the Russian Foundation for Basic Research (grant 14-01-00628).

1. Introduction

The structure of the article is as follows. In the section 2 basic notations and conventions are introduced. The common ideology of adaptive congestion control and RED mechanism are presented in the section 3.1. The method of stochastization of one-step processes is applied in order to construct the model of RED module. The sense of this method and its components are described in section 4.

Next, the method of construction of the one-step processes (RED, in particular) is described. The presentation is based on the following pattern. At the beginning of the section the general theoretical information is given. At the end of the section the considered theoretical constructs are directly applied to the RED modeling. The master equation overview is presented in the section 5. The interaction schemes are introduced in the section 6. The combinatorial method of modeling is discussed in the section 7.

The operator model approach is presented in the section 8 where, in particular, the algorithm of transition to the occupation number representation is described. Also the master equation in the form of the Liouville operator equation is written down.

2. Notations and conventions

1. The abstract indices notation [1] is used in this work. Under this notation a tensor as a whole object is denoted just as an index (e.g., x^i), components are denoted by underlined index (e.g., $x^{\underline{i}}$).

2. We will adhere to the following agreements. Latin indices from the middle of the alphabet (i, j, k) will be applied to the space of the system state vectors. Latin indices from the beginning of the alphabet (a) will be related to the Wiener process space. Greek indices (α) will set a number of different interactions in kinetic equations.

3. Adaptive congestion management

3.1 RED Adaptive congestion control mechanism

To improve the channel performance the queue management on routers needs to be optimized. One of the possible approaches is to use the random early detection algorithm (Random Early Detection, RED) [2] or RED modifications [3].

The module that implements the RED-type algorithm can be schematically represented as follows (fig. 1)

RED algorithm uses a weighted value of the queue length $\hat{Q}$ as factor for packet dropping risk determination. In order to calculate $\hat{Q}$ the exponentially weighted moving-average (EWMA) is used:

$$\hat{Q}_{k+1} = (1 - w_q)\hat{Q}_k + w_q Q_k, \quad k = 0, 1, 2, \ldots,$$

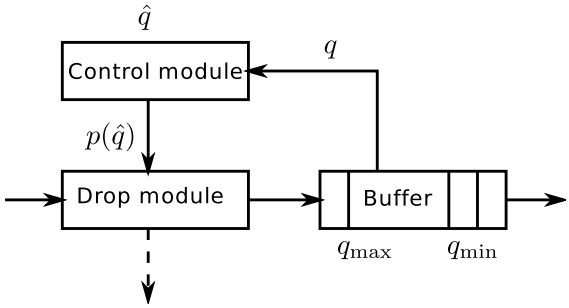

Figure 1: RED module

where w_q, $0 < w_q < 1$ — a weight coefficient of the exponentially weighted moving-average.

As the the average queue length increases, the packets drop probability also increases (see. (1)). For dropping management algorithm the dropping function with two average queue length thresholds $Q_{\min}$ and $Q_{\max}$ as arguments (Fig. 2) is used:

$$p(\hat{Q}) = \begin{cases} 0, & 0 < \hat{Q} \leqslant Q_{\min}, \\ \dfrac{\hat{Q} - Q_{\min}}{Q_{\max} - Q_{\min}} p_{\max}, & Q_{\min} < \hat{Q} \leqslant Q_{\max}, \\ 1, & \hat{Q} > Q_{\max}. \end{cases} \tag{1}$$

Here $p(\hat{Q})$ — package dropping function, $\hat{Q}$ — the queue length weighted average, $Q_{\min}$ and $Q_{\max}$ — thresholds of queue length weighted average, $p_{\max}$ — the maximum level of packages reset.

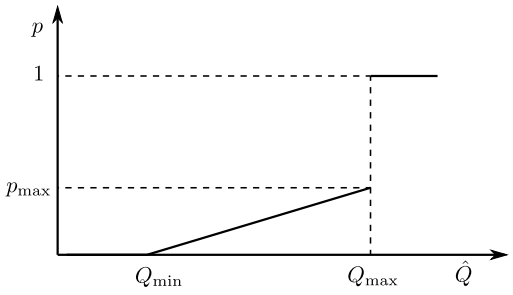

Figure 2: RED drop function

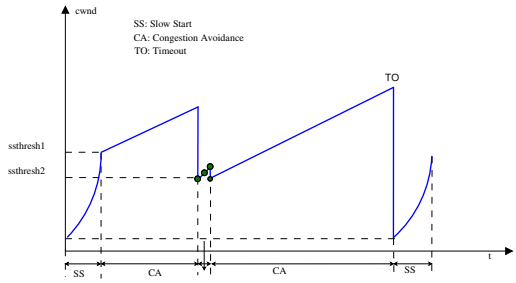

Figure 3: TCP phases

3.2 TCP congestion control mechanism

TCP protocol uses a moving mechanism to overcome network traffic congestion. The implementation of this mechanism depends on the specific standard of TCP protocol . Following the original articles [4, 5] we consider the TCP Reno.

The TCP Reno traffic congestion control mechanism consists of the following phases: slow start, congestion avoidance, fast transmission and fast recovery. Dy-

namics of changes of the congestion window size (Congestion Window, CWND) depends on the phase (see. Fig. 3).

In the phase of a slow start, there is an increase of congestion window whenever a source receives a delivery notification packet (Acknowledge, ACK), i.e. source increases the congestion window size depending on the quantity of confirmed segments (Segment Size, SS): $cwnd = cwnd + 1$ for each transmitted ACK. Initial congestion window size (Minimum Segment Size, MSS) can be set to 1, 2 or 10 segments. Receiver sends an ACK for each packet, but in practice it can be assumed that confirmation come together after a time of double circulation (Round-Trip Time, RTT). Therefore the congestion window is doubled after a time of double turn.

For a certain window size TCP Reno enters in the congestion avoidance phase. In this phase, the congestion window is increased by the amount of $1/cwnd$ for each acknowledgment ACK, which is equivalent to increasing of the window by one packet for the double-turn.

Protocol TCP Reno monitors two options of packet loss:

- Triple Duplicate ACK (TD). Let n-th package is not delivered, and subsequent packets ($n + 1$, $n + 2$, etc.) are delivered. For each delivered in violation prioritization packet,(for $n + 1$, $n + 2$, and so on) the recipient sends ACK message for the last undelivered (n-th) package. With receiving three such packets the source resends the n-th package. In addition, the window size is decreased by 2 times $cwnd \to cwnd/2$.

- Timeout (TO). When sending a package the timeout timer is started. When the confirmation is received the timer is restarted. Wherein the window size is set to the initial value of the congestion window. The first lost package is resent. The protocol proceeds in a slow start phase.

Overall congestion control algorithm belongs to AIMD algorithms type (Additive Increase, Multiplicative Decrease) — an additive increase of the window size and multiplicative decrease of it.

3.3 The construction of the RED module model

Let's consider the model containing two basic elements: the source and the recipient. As the recipient we use the queue. A source sends packets, the receiver processes them and sends an acknowledgment of the adoption of the package.

In constructing the model, we start from the idea that the source and the recipient does not directly interact. The interaction occurs by management. In connection with this we obtain the two one-dimensional equations, the first describes the TCP window, and another — instantaneous queue length. For the source we prescribe sending packets intensity, which depends on the size of the window. The source and the recipient interact through management, i.e., through the queue influence on source parameters.

Because our model is the realization of the existing model based on the new ideology, then for the implementation we will use the same model assumptions that in previous works [6, 7, 8]. Also we note that our task is not a complete duplication of the previous model but a demonstration of the applicability of stochastization of single-step processes for traffic control models. In this way:

- we consider only the overload prevent stage;
- we will take into account the loss of packets only because of triple duplicate confirmation that will not obscure our model by details and gives us the opportunity to demonstrate the methodology more clearly.

4. General review of the methodology

The interaction schemes are the starting point of our methodology [1] [9, 10]. Any scheme of interaction unambiguously corresponds to a master equation [11, 12], but master equation is usually quite complicated and it's make it difficult for study and sollving. Our technique involves two possibilities (see Fig. 4):

- computational approach; the solution of the master equation with help of perturbation theory;
- modeling approach; to obtain approximate models in the form Fokker–Planck and Langevin equations.

The model approach provides a model that is convenient to study with numerical and qualitative approaches. In addition, this approach assumes the iterative process of research: the obtained approximate model can be specified and changed, which leads to the correction of initial interaction schemes.

In this article we are interested in the model approach.

There are two ways of building the master equation[2]

- combinatorial approach (see Fig. 5);
- operatorial approach (see Fig. 6).

[1] The analogs of the interaction schemes are the equations of chemical kinetics, reaction particles and etc.

[2] In quantum field theory the path integrals approach can be considered as an analogue of the combinatorial approach and the method of second quantization as analog of the operational approach

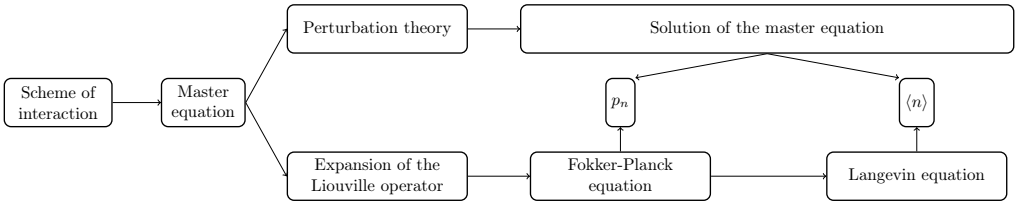

Figure 4: The general structure of the methodology

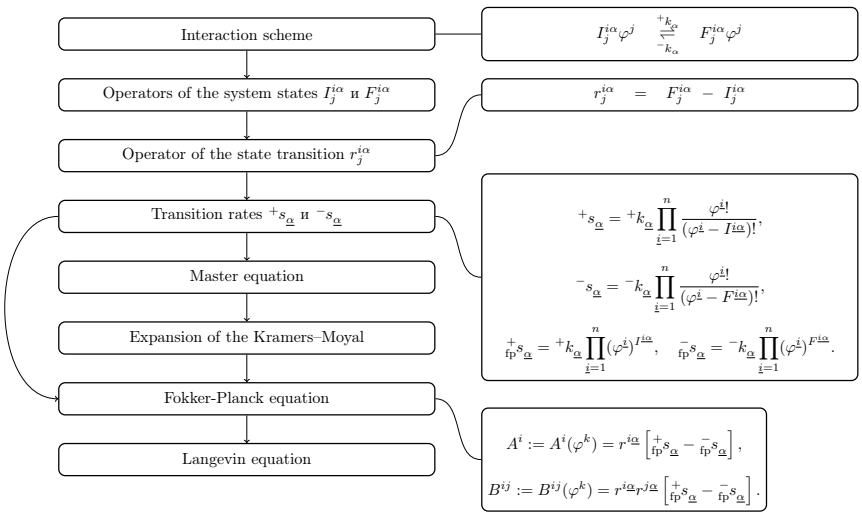

Figure 5: Combinatorial modeling approach

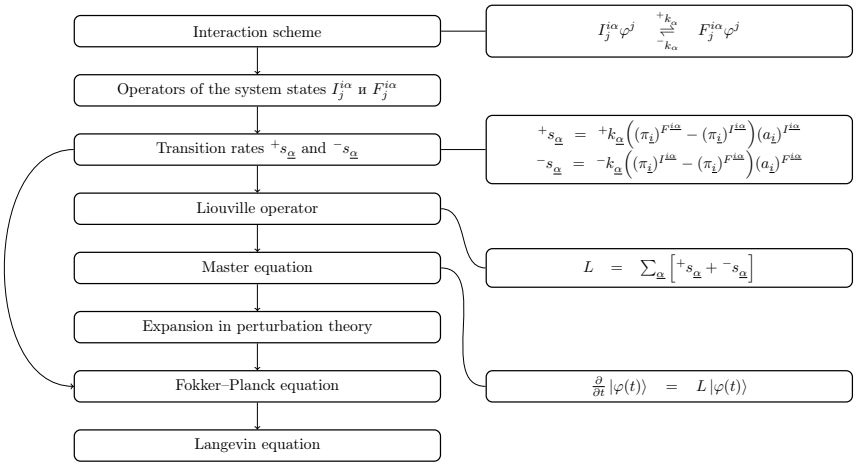

Figure 6: Operational modeling approach

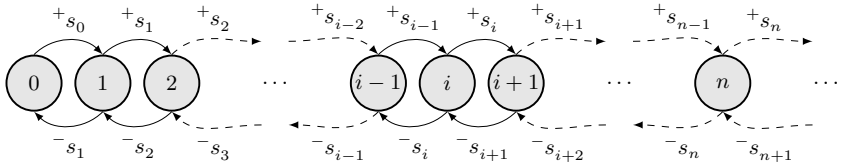

Figure 7: One–step process

5. The master equation

For the system description we will use the master equation, which describes the transition probabilities for Markov process [11, 12]:

$$\frac{\partial p(\varphi_2, t_2|\varphi_1, t_1)}{\partial t} = \int \left[w(\varphi_2|\psi, t_2) p(\psi, t_2|\varphi_1, t_1) - w(\psi|\varphi_2, t_2) p(\varphi_2, t_2|\varphi_1, t_1) \right] \mathrm{d}\psi \,,$$

where $w(\varphi|\psi, t)$ is the probability of transition from the state ψ to the state φ for unit time.

Fixing the initial values of φ_1, t_1, we can write the equation for subensemble:

$$\frac{\partial p(\varphi, t)}{\partial t} = \int \left[w(\varphi|\psi, t) p(\psi, t) - w(\psi|\varphi, t) p(\varphi, t) \right] \mathrm{d}\psi \,. \tag{2}$$

If a domain of φ is a discrete one then the (2) can be written as follows (the states are numbered by n and m):

$$\frac{\partial p_n(t)}{\partial t} = \sum_m \left[w_{nm} p_m(t) - w_{mn} p_n(t) \right], \tag{3}$$

where the p_n is the probability of the system to be in a state n at time t, w_{nm} is the probability of transition from the state m in the state n per unit time.

There are two types of system transition from one state to another (based on one–step processes) as a result of system elements interaction: in the forward direction $(\varphi^i + r_j^{i\underline{\alpha}} \varphi^j)$ with the probability ${}^{+}s_{\underline{\alpha}}(\varphi^k)$ and in the opposite direction $(\varphi^i - r_j^{i\underline{\alpha}} \varphi^j)$ with the probability ${}^{-}s_{\underline{\alpha}}(\varphi^k)$ (fig. 7). The matrix of transition probabilities has the form:

$$w_{\underline{\alpha}}(\varphi^i|\psi^i, t) = {}^{+}s_{\underline{\alpha}} \delta_{\varphi^i, \psi^i + 1} + {}^{-}s_{\underline{\alpha}} \delta_{\varphi^i, \psi^i - 1}, \qquad \underline{\alpha} = \overline{1, s},$$

where $\delta_{i,j}$ is Kronecker delta.

Thus, the general form of the master equation for the state vector φ^i, changing by steps with length $r_j^{i\underline{\alpha}} \varphi^j$, is:

$$\frac{\partial p(\varphi^i, t)}{\partial t} = \sum_{\underline{\alpha}=1}^{s} \Big\{ {}^{-}s_{\underline{\alpha}}(\varphi^i + r^{i\underline{\alpha}}, t) p(\varphi^i + r^{i\underline{\alpha}}, t) + {}^{+}s_{\underline{\alpha}}(\varphi^i - r^{i\underline{\alpha}}, t) p(\varphi^i - r^{i\underline{\alpha}}, t) - \\ - \Big[s_{\underline{\alpha}}(\varphi^i) + {}^{-}s_{\underline{\alpha}}(\varphi^i) \Big] p(\varphi^i, t) \Big\} . \tag{4}$$

6. Interaction schemes

6.1 General description

The system state is defined by the vector $\varphi^i \in \mathbb{R}^n$, where n — is system order [1]. The operator $I_j^i \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ describes the state of the system before the interaction, the operator $F_j^i \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ — after the interaction[2]. The result of interaction is the system transition from one state to another one.

There are s types of interaction in our system, so instead of I_j^i and F_j^i operators we will use operators $I_j^{i\underline{\alpha}} \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_+^s$ and $F_j^{i\underline{\alpha}} \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_+^s$ [1].

The interaction of the system elements will be described by interaction schemes, which are similar to schemes of chemical kinetics:

$$I_j^{i\underline{\alpha}}\varphi^j \underset{{}^{-}k_{\underline{\alpha}}}{\overset{{}^{+}k_{\underline{\alpha}}}{\rightleftharpoons}} F_j^{i\underline{\alpha}}\varphi^j, \qquad \underline{\alpha} = \overline{1,s}, \tag{5}$$

the Greek indices specify the number of interactions and Latin are the system order. The coefficients ${}^{+}k_{\underline{\alpha}}$ and ${}^{-}k_{\underline{\alpha}}$ have meaning intensity (speed) of interaction.

The state transition is given by the operator:

$$r_j^{i\underline{\alpha}} = F_j^{i\underline{\alpha}} - I_j^{i\underline{\alpha}}. \tag{6}$$

Thus, one step interaction $\underline{\alpha}$ in forward and reverse directions can be written as

$$\varphi^i \to \varphi^i + r_j^{i\underline{\alpha}}\varphi^j,$$
$$\varphi^i \to \varphi^i - r_j^{i\underline{\alpha}}\varphi^j.$$

We can also write (5) not in the form of vector equations but in the form of sums:

$$I_j^{i\underline{\alpha}}\varphi^j\delta_i \underset{{}^{-}k_{\underline{\alpha}}}{\overset{{}^{+}k_{\underline{\alpha}}}{\rightleftharpoons}} F_j^{i\underline{\alpha}}\varphi^j\delta_i,$$

where $\delta_{\underline{i}} = (1, \ldots, 1)$.

Also the following notation will be used:

$$I^{i\underline{\alpha}} := I_j^{i\underline{\alpha}}\delta^j, \quad F^{i\underline{\alpha}} := F_j^{i\underline{\alpha}}\delta^j, \quad r^{i\underline{\alpha}} := r_j^{i\underline{\alpha}}\delta^j.$$

6.2 The interaction schemes for RED model

For RED model construction we will use two variables:

- W (congestion window) is a characteristic of the source;

[1] For brevity, we denote the module over the field $\mathbb{R}$ just as $\mathbb{R}$.

[2] The component dimension indices take on values $\underline{i}, \underline{j} = \overline{1,n}$

[1] The component indices of number of interactions take on values $\underline{\alpha} = \overline{1,s}$

- Q (instantaneous queue length) is a characteristic of maintenance devices.

Furthermore, since the dropping function of RED module depends on exponentially weighted moving average queue length, we will need an equation of $\hat{Q}$ and Q, where $\hat{Q}$ has an exponentially weighted moving average of the instant queue length Q.

There is a management of interactions in the RED model. Thus patterns of interaction will be recorded for each variable separately.

The equation for the congestion window

Let's consider the behavior of packages in the system. Number of packages will specify the size of the TCP window. There are two processes: with the intensity $1/T$ packets from infinite reservoir fed into the system, and with an intensity $1/2\,(\mathrm{d}N/\mathrm{d}t)$ packets leave the system. Here $\mathrm{d}N$ — the Poisson process, similar to the entered in [5]. We write the appropriate schemes of interaction:

$$0 \underset{1/2\,(\mathrm{d}N/\mathrm{d}t)}{\overset{1/T}{\rightleftharpoons}} W. \tag{7}$$

The direct scheme describes the appearance of packages in the system, the indirect one — output of packages from the system.

The equation for the instantaneous queue length

The behavior of the queue is described in similar way. There are two processes: with the intensity W/T packets are received from by the queue the infinite reservoir, and with intensity C packets leave the system. This can be written in the form of kinetic equations:

$$\begin{cases} 0 \xrightarrow{W/T} Q, \\ 0 \xrightarrow{-C} Q. \end{cases} \tag{8}$$

7. Combinatorial approach

We will obtain the function ${}^{+}s_{\underline{\alpha}}$ and ${}^{-}s_{\underline{\alpha}}$ for equation (4) with use of combinatorial approach.

7.1 The transition probabilities

The transition rates ${}^{+}s_{\underline{\alpha}}$ and ${}^{-}s_{\underline{\alpha}}$ are proportional to the number of ways of choosing the number of arrangements of $\varphi^{\underline{i}}$ to $I^{\underline{i\alpha}}$ (denoted as $A^{I^{\underline{i\alpha}}}_{\varphi^{\underline{i}}}$) and to $F^{\underline{i\alpha}}$ (denoted as

$A_{\underline{\varphi^i}}^{F^{\underline{i\alpha}}}$) and defined by:

$$
\begin{aligned}
{}^{+}s_{\underline{\alpha}} &= {}^{+}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} A_{\underline{\varphi^i}}^{I^{\underline{i\alpha}}} = {}^{+}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} \frac{\underline{\varphi^i}!}{(\underline{\varphi^i} - I^{\underline{i\alpha}})!}, \\
{}^{-}s_{\underline{\alpha}} &= {}^{-}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} A_{\underline{\varphi^i}}^{F^{\underline{i\alpha}}} = {}^{-}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} \frac{\underline{\varphi^i}!}{(\underline{\varphi^i} - F^{\underline{i\alpha}})!}.
\end{aligned}
\tag{9}
$$

Replacing in (9) the $\varphi(\varphi-1)\cdots(\varphi-(n-1))$-type combinations on $(\varphi)^n$ we obtain for Fokker–Planck equation[1]:

$$
\begin{aligned}
{}^{+}_{\mathrm{fp}}s_{\underline{\alpha}} &= {}^{+}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} (\underline{\varphi^i})^{I^{\underline{i\alpha}}}, \\
{}^{-}_{\mathrm{fp}}s_{\underline{\alpha}} &= {}^{-}k_{\underline{\alpha}} \prod_{\underline{i}=1}^{n} (\underline{\varphi^i})^{F^{\underline{i\alpha}}}.
\end{aligned}
\tag{10}
$$

7.2 Fokker–Planck equation

Fokker–Planck equation is a special case of the master equation and can be regarded as its approximation. We can get through the expansion of the master equation in a series up to the second order. We will use the decomposition of the Kramers–Moyal [12] (for simplicity it is written for the one-dimensional case):

$$
\frac{\partial p(\varphi,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \varphi^n} \Big[\xi^n(\varphi) p(\varphi,t) \Big],
$$

where

$$
\xi^n(\varphi) = \int_{-\infty}^{\infty} (\psi - \varphi)^n w(\psi|\varphi)\, \mathrm{d}\psi\,.
$$

By dropping the terms with order higher than the second one, we obtain the Fokker–Planck equation:

$$
\frac{\partial p(\varphi,t)}{\partial t} = -\frac{\partial}{\partial \varphi} \left[A(\varphi) p(\varphi,t) \right] + \frac{\partial^2}{\partial \varphi^2} \left[B(\varphi) p(\varphi,t) \right],
$$

and for multivariate case

$$
\frac{\partial p(\varphi^k,t)}{\partial t} = -\frac{\partial}{\partial \varphi^i} \left[A^i(\varphi^k) p(\varphi^k,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial \varphi^i \partial \varphi^j} \left[B^{ij}(\varphi^k) p(\varphi^k,t) \right], \tag{11}
$$

where

$$
\begin{aligned}
A^i &:= A^i(\varphi^k) = r^{i\underline{\alpha}} \Big[s_{\underline{\alpha}} - {}^{-}_{\mathrm{fp}}s_{\underline{\alpha}} \Big], \\
B^{ij} &:= B^{ij}(\varphi^k) = r^{i\underline{\alpha}} r^{j\underline{\alpha}} \Big[s_{\underline{\alpha}} - {}^{-}_{\mathrm{fp}}s_{\underline{\alpha}} \Big].
\end{aligned}
\tag{12}
$$

[1] This change corresponds to a series expansion.

As can be seen from the (12), the coefficients of the Fokker–Planck equation can be obtained directly from the (6) and (9), that is, in this case, it is not necessary to write down the master equation.

7.3 Langevin equation

The Langevin equation which corresponds to Fokker–Planck equation:

$$\mathrm{d}\varphi^i = a^i\,\mathrm{d}t + b^i_a\,\mathrm{d}W^a\,, \tag{13}$$

where $a^i := a^i(\varphi^k)$, $b^i_a := b^i_a(\varphi^k)$, $\varphi^i \in \mathbb{R}^n$ — system state vector, $W^a \in \mathbb{R}^m$ — m-dimensional Wiener process[1]. Latin indices from the middle of the alphabet will be applied to the system state vectors (the dimensionality of space is n), and Latin indices from the beginning of the alphabet denote the variables related to the Wiener process vector (he dimensionality of space is $m \leqslant n$).

The connection between the equations (11) and (13) is expressed by the following relationships:

$$A^i = a^i, \qquad B^{ij} = b^i_a b^{ja}.$$

7.4 A combinatorial approach in the application to the RED model

The congestion window equation

An equation for the interaction scheme (7). The number of packets before and after the interaction, and the transition operator are:

$$I^{\underline{\alpha}1} = \big(0\big)\,, \qquad F^{\underline{\alpha}1} = \big(1\big)\,, \qquad r^{\underline{\alpha}1} = \big(1\big)\,. \tag{14}$$

With respect of (9), (10) and (14), we get

$${}^{+}s_1 = {}^{+}_{\mathrm{fp}}s_1 = \frac{1}{T}, \qquad {}^{-}s_1 = {}^{-}_{\mathrm{fp}}s_1 = \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}W. \tag{15}$$

With use of (3), (4) and (15) we can get the master equation:

$$\frac{\partial W_n(t)}{\partial t} = \frac{n+1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}W_{n+1}(t) + \frac{1}{T}W_{n-1}(t) - \left[\frac{1}{T} + \frac{n}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\right]W_n(t). \tag{16}$$

where $W_n(t)$ — is probability that window size is n at the moment t.

Using (12) and (15), we will get Fokker-Planck equation coefficients:

$$A^i = \frac{1}{T} - \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}W,$$
$$B^{ij} = \frac{1}{T} - \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}W.$$

[1]Wiener process is realized as $\mathrm{d}W = \varepsilon\sqrt{\mathrm{d}t}$, where $\varepsilon \sim N(0,1)$ — the normal distribution with mean 0 and variation 1.

The Fokker-Planck equation:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial W}\left[\left(\frac{1}{T} - \frac{W}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\right)w\right] + \frac{1}{2}\frac{\partial^2}{\partial W^2}\left[\left(\frac{1}{T} - \frac{W}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\right)w\right],$$

where $w := w(t)$ — the distribution density of a random process $W(t)$.

The corresponding Langevin equation is:

$$\mathrm{d}W = \frac{1}{T}\,\mathrm{d}t - \frac{W}{2}\,\mathrm{d}N + \sqrt{\frac{1}{T} - \frac{W}{2}\frac{\mathrm{d}N}{\mathrm{d}t}}\,\mathrm{d}V^1\,,$$

where $\mathrm{d}V^1$ — Wiener process, which corresponds to the random process $W(t)$.

The equation for the instantaneous queue length

Now, we will obtain the equation for the interaction scheme (8):

$$I^{\underline{\alpha 1}} = \begin{pmatrix}0\\0\end{pmatrix}, \quad F^{\underline{\alpha 1}} = \begin{pmatrix}1\\1\end{pmatrix}, \quad r^{\underline{\alpha 1}} = \begin{pmatrix}1\\1\end{pmatrix}. \tag{17}$$

Taking into account (9), (10) and (17), we will get:

$${}^{+}s_1 = {}^{+}_{\mathrm{fp}}s_1 = \frac{W}{T}, \qquad {}^{+}s_2 = {}^{+}_{\mathrm{fp}}s_2 = -C. \tag{18}$$

Based on (3), (4) and (18) we will write the master equation:

$$\frac{\partial Q_n(t)}{\partial t} = \left[\frac{W}{T} - C\right]Q_{n-1}(t) - \left[\frac{W}{T} - C\right]Q_n(t).$$

where $Q_n(t)$ is the probability that queue length is n at the moment t.

From (12) and (9), we get Fokker–Planck equation coefficients:

$$A^i = \frac{W}{T} - C,$$
$$B^{ij} = \frac{W}{T} - C.$$

The Fokker–Planck equation:

$$\frac{\partial q}{\partial t} = -\frac{\partial}{\partial Q}\left[\left(\frac{W}{T} - C\right)q\right] + \frac{1}{2}\frac{\partial^2}{\partial Q^2}\left[\left(\frac{W}{T} - C\right)q\right],$$

where $q := q(t)$ — the distribution density of a random process Q. Then Langevin equation is:

$$\mathrm{d}Q = \left(\frac{W}{T} - C\right)\mathrm{d}t + \sqrt{\frac{W}{T} - C}\,\mathrm{d}V^2\,,$$

where $\mathrm{d}V^2$ is a Wiener process, which corresponds to a random process Q.

8. Operatorial approach

8.1 Occupation numbers representation

Occupation number representation is the main language in the description of many-body physics. The main elements of the language are the wave functions of the system, providing information about how many particles are in each single-particle state. The creation and annihilation operators are used for system states change. The advantages of this formalism are following:

- it is possible to consider systems with a variable number of particles (non-stationary systems);
- system statistics (Fermi–Dirac or Bose–Einstein) is automatically included in the commutation rules for the creation and annihilation operators;
- this is the second major formalism (along with the path integral) for the quantum perturbation theory description.

The method of application of the formalism of second quantization for the non-quantum systems (statistical, deterministic systems) was studied in a series of articles [13, 14, 15, 16].

The Dirac notation is commonly used for occupation numbers representation recording.

8.2 Dirac notation

This notation is proposed by P. A. M. Dirac [17][1]. Under this notation, the vector φ^i is defined as $|i\rangle$, and covariant vector (covector) φ_i is defined as $\langle i|$. Conjunction operation is used for raising and lowering of indices[2]:

$$\varphi_i^* := \varphi_i = (\varphi^i)^\dagger \equiv \langle i| = |i\rangle^\dagger .$$

The scalar product is as follows:

$$\varphi_i \varphi^i \equiv \langle i|i\rangle .$$

The tensor product is:

$$\varphi_j \varphi^i \equiv |i\rangle \langle j| .$$

Furthermore, this form of notation is also possible:

$$|\varphi\rangle := \varphi^i, \qquad \langle i|\varphi\rangle := \varphi^i \delta_i^{\underline{i}} = \varphi^{\underline{i}}.$$

[1] The notation is based on the notation, proposed by G. Grassmann in 1862 [18, p. 134].

[2] In this case, we use Hermitian conjugation $\bullet^\dagger$. The sign of the complex conjugate $\bullet^*$ in this entry is superfluous.

8.3 Birth and death operators

The transition to the space of occupation numbers is not a unitary transformation. However, the algorithm of transition (specific to each task) can be constructed.

Let's write the master equation (3) in the occupation number representation. We will consider a system that does not depend on the spatial variables. For simplicity, we consider the one-dimensional version.

Let's denote in (3) the probability that there are n particles in our system as φ_n:

$$\varphi_n := p_n(\varphi, t).$$

The vector space $\mathcal{H}$ consists of states of φ.

The scalar product:

$$\langle\varphi|\psi\rangle = \sum_n n! p_n^*(\varphi) p^n(\psi) = \sum_n n! \varphi_n^*(\varphi) \varphi^n(\psi) \tag{19}$$

and $|n\rangle$ — basis vectors.

From $p_n(m) = \delta_n^m$ and (19) we can obtain:

$$\langle n|m\rangle = n! \delta_n^m. \tag{20}$$

The state vector:

$$|\varphi\rangle = \sum_n p_n(\varphi) |n\rangle = \sum_n \varphi_n |n\rangle =: \varphi_n |n\rangle . \tag{21}$$

In view of (20) the following expression may be written:

$$\varphi_n = \frac{1}{n!} \langle n|\varphi\rangle . \tag{22}$$

Let's use birth and death operators:

$$\begin{aligned} \pi |n\rangle &= |n+1\rangle , \\ a |n\rangle &= n |n-1\rangle \end{aligned} \tag{23}$$

and commutation rule[1]:

$$[a, \pi] = 1. \tag{24}$$

If the form of scalar product is (19) then with the help of (24) it is obviously that the our system is described by Bose–Einstein statistics.

From the relation (20) we obtain:

$$\langle m|a^\dagger|n\rangle = \langle m|\pi|n\rangle ,$$

and for the scalar product (19) the following statement is valid:

$$a^\dagger = \pi.$$

[1] In fact, $a\pi |n\rangle - \pi a |n\rangle = (n+1) |n\rangle - n |n\rangle = |n\rangle$.

8.4 Liouville operator

The Liouville equation:

$$\frac{\partial}{\partial t}\left|\varphi(t)\right\rangle = L\left|\varphi(t)\right\rangle. \tag{25}$$

Liouville operator L satisfies the relation:

$$\langle 0|\, L = 0.$$

From (3), (21), (22) and (25) we obtain:

$$\frac{\partial p_n}{\partial t} = \frac{1}{n!}\left\langle n\left|\frac{\partial}{\partial t}\right|\varphi\right\rangle = \frac{1}{n!}\langle n|L|\varphi\rangle = \sum_m [w_{nm}p_m - w_{mn}p_n], \tag{26}$$

The Liouville equation (25) in the form of a single equation writes down master equations (3) for different values of n.

The following Liouville operator corresponds to the scheme (5):

$$L = \sum_{\underline{\alpha},\underline{i}} \left[k_{\underline{\alpha}}\Big(()^{F^{\underline{i\alpha}}} - (\pi_{\underline{i}})^{I^{\underline{i\alpha}}}\Big)(a_{\underline{i}})^{I^{\underline{i\alpha}}} + {}^{-}k_{\underline{\alpha}}\Big(()^{I^{\underline{i\alpha}}} - (\pi_{\underline{i}})^{F^{\underline{i\alpha}}}\Big)(a_{\underline{i}})^{F^{\underline{i\alpha}}}\right]. \tag{27}$$

8.5 Operator approach in the application for the RED model

The equation for the congestion window

As the basis vectors we consider the size of the window $|{}_w n\rangle$ (or simple $|n\rangle$), if the basis choice is clear from the context. Consider W_n as probability that W has size n.

$$\langle W_1|W_2\rangle = \sum_n n! W_n^*(W_1) W^n(W_2).$$

Taking into account the normalization (20) and (22) we will write

$$\langle n|W\rangle = n! W_n. \tag{28}$$

As in (23) we introduce birth–death operators

$${}_w a^\dagger\,|n\rangle =: a^\dagger\,|n\rangle = |n+1\rangle\,,$$
$${}_w a\,|n\rangle =: a\,|n\rangle = n\,|n-1\rangle$$

with commutator

$$\left[a,\, {}_w a^\dagger\right] = 1.$$

Based on (7) and (27) we can write Liouville operator for the congestion window:

$${}_w L = \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\,{}_w a + \frac{1}{T}\,{}_w a^\dagger - \frac{1}{T} - \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\,{}_w a^\dagger\,{}_w a. \tag{29}$$

With use of (26), (28), we can show, that (29) is eq (16):

$$\frac{\partial W_n(t)}{\partial t} = \frac{1}{n!}\left\langle n\left|\frac{\partial}{\partial t}\right|W(t)\right\rangle = \frac{1}{n!}\langle n|_w L|W(t)\rangle =$$
$$= \frac{1}{n!}\left\langle n\left|\frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}{}_w a + \frac{1}{T}{}_w a^\dagger - \frac{1}{T} - \frac{1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}{}_w a^\dagger{}_w a\right|W(t)\right\rangle =$$
$$= \frac{n+1}{2}\frac{\mathrm{d}N}{\mathrm{d}t}W_{n+1}(t) + \frac{1}{T}W_{n-1}(t) - \left[\frac{1}{T} + \frac{n}{2}\frac{\mathrm{d}N}{\mathrm{d}t}\right]W_n(t).$$

The equation for the instantaneous queue length

As the basis vectors we consider the size of the window $\left|{}_q n\right\rangle$ or simple $|n\rangle$, if the basis choice is clear from the context. Consider Q_n as probability that Q has size n.

$$\langle Q_1|Q_2\rangle = \sum_n n! Q_n^*(Q_1)Q^n(Q_2).$$

Taking into account the normalization (20) and (22) we will write

$$\langle n|Q\rangle = n!Q_n.$$

As in (23) we introduce birth–death operators

$${}_q a^\dagger\,|n\rangle =: a^\dagger\,|n\rangle = |n+1\rangle\,,$$
$${}_q a\,|n\rangle =: a\,|n\rangle = n\,|n-1\rangle$$

with commutator:

$$\left[a, {}_q a^\dagger\right] = 1.$$

Based on (7) and (27) we can write Liouville operator for the congestion window:

$${}_q L = \frac{W}{T}({}_q a - 1) - C({}_q a - 1).$$

With use of (26), (28), we can show, that (29) is eq (16):

$$\frac{\partial Q_n(t)}{\partial t} = \frac{1}{n!}\left\langle n\left|\frac{\partial}{\partial t}\right|Q(t)\right\rangle = \frac{1}{n!}\langle n|{}_q L|Q(t)\rangle =$$
$$= \frac{1}{n!}\left\langle n\left|\frac{W}{T}({}_q a - 1) - C({}_q a - 1)\right|W(t)\right\rangle =$$
$$= \left[\frac{W}{T} - C\right]Q_{n-1}(t) - \left[\frac{W}{T} - C\right]Q_n(t).$$

9. Conclusions

To construct the model of the RED control module the method of stochastization of one-step processes in the representation of state vectors (combinatorial method) and the operator form was applied. At all stages the operator method is compared

with the combinatorial method of stochastization of one-step processes. The logic of both methods is demonstrated. The comparison of both methods shows their complete equivalence.

In addition, the construction of the stochastic model of the RED control module coincides with the previously developed deterministic model. Thus, the effectiveness of one-step processes stochastization method is demonstrated.

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