http://mmg.tversu.ru ISSN 2311-1275

MMG *Mathematical Modelling and Geometry*

Volume 3, No 1, p. 43 – 63 (2015)

Waveguide modes of a planar optical waveguide

M. N. Gevorkyan[1,a], **D. S. Kulyabov**[1,2,b], **K. P. Lovetskiy**[1,c], **A. L. Sevastyanov**[1,d], and **L. A. Sevastyanov**[1,3,e]

[1] Department of Applied Probability and Informatics, Peoples' Friendship University of Russia, Miklukho-Maklaya str. 6, 117198 Moscow, Russia

[2] Laboratory of Information Technologies, Joint Institute for Nuclear Research, Joliot-Curie 6, 141980 Dubna, Moscow region, Russia

[3] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Joliot-Curie 6, 141980 Dubna, Moscow region, Russia

e-mail: [a] mngevorkyan@sci.pfu.edu.ru, [b] dharma@sci.pfu.edu.ru, [c] lovetskiy@gmail.com, [d] alsevastyanov@gmail.com, [e] leonid.sevast@gmail.com

Received 25 January 2015, in final form 11 February. Published 12 February 2015.

The work partially supported by RFBR grants Nos 13-01-00595, 14-01-00628, and 15-07-08795.

Abstract. In a planar regular optical waveguide, propagation of polarized monochromatic electromagnetic radiation obeys a law following from the Maxwell equations. The Maxwell equations in Cartesian coordinates associated with the waveguide geometry can be written as the two independent systems of equations:

$$E_x = \frac{\beta}{\varepsilon} H_y,\ \frac{dE_z}{dx} = \frac{ik_0}{\varepsilon}\left(\varepsilon\mu - \beta^2\right) H_y,\ \frac{dH_y}{dx} = ik_0 \varepsilon E_z,$$

$$H_x = -\frac{\beta}{\mu} E_y,\ \frac{dH_z}{dx} = -\frac{ik_0}{\mu}\left(\varepsilon\mu - \beta^2\right) E_y,\ \frac{dE_y}{dx} = -ik_0 \mu H_z.$$

Each of the systems can be transformed to a second order ODE for the leading component and two other equations for straightforward computation of the complementary electromagnetic field components. In doing so, the boundary conditions for Maxwell's equations are reduced to two pairs of boundary conditions for obtained equations. In addition, the asymptotic conditions hold for each class of waveguide modes. Thus, the problem of description of a complete set of modes in a regular planar waveguide is formulated in terms of the eigenvalues problem for the essentially self-adjoint second order differential operator:

$$-\frac{d^2\psi}{dx^2} + V(x)\,\psi = k^2\psi.$$

For the operator, we find some results about its spectrum, complete sets of solutions, and diagonalization by an isometric isomorphism (generalized Fourier transformation); new basis functions are related to initial ones by simple transformation formulas. The eigenvalues problem is equivalently reduced to the two problems (left and right) of the one-dimensional potential scattering theory by projection on the two branches of the continuous spectrum.

Keywords: waveguide propagation of electromagnetic radiation, equations of waveguide modes of regular waveguide, guided modes, radiation modes, a complete set of modes of a planar waveguide.

MSC numbers: 65Fxx, 65Hxx, 65L10, 65L15, 78A40, 78Mxx

1. Introduction

To describe propagation of electromagnetic radiation in integrated optical waveguides by the coupled-wave method [1, 2], by the comparison-of-waveguides method [3, 4], or by the incomplete Galerkin method [5, 6], we need to know a complete system of waveguide modes of a regular planar waveguide [7, 8] and be able to work with them. In this work we consider the special, but the most widespread case of a multilayer waveguide.

There are the three types of waveguide modes in a regular planar optical waveguide: guided modes, substrate radiation modes, and cover radiation modes. The regular waveguide consists of a dielectric waveguide layer (or a few ones) of refractive index n_f (or n_{f1}, ..., n_{fN}) and the dielectric cladding with smaller refraction indices: n_s in the substrate layer and n_c in the cover layer. We will use Cartesian coordinates associated with the waveguide geometry. The waveguide layer thickness, say d, is about of the monochromatic electromagnetic radiation wavelength, while thicknesses of the substrate and cover layer are supposed to be much greater and, in our model, will be considered as infinite quantities.

The mathematical model of light propagation in a waveguide consists of the Maxwell equations supplemented by the matter equations and boundary conditions. In the coordinates adapted to the waveguide geometry as in Figure 1, the Maxwell equations can be split into two independent sets for the TE and TM polarizations. Their solutions are, respectively,

$$E_y\left(x, y, z, t\right) = E_y\left(x\right) \exp\left\{i\omega t - i\beta z\right\}$$

and

$$H_y\left(x, y, z, t\right) = H_y\left(x\right) \exp\left\{i\omega t - i\beta z\right\},$$

where ω is the angular frequency, β is the phase delay coefficient of the waveguide mode, x, y, z are space dimensionless coordinates, and the functions $E_y\left(x\right)$ and $H_y\left(x\right)$ for TE and TM modes, respectively, are determined by the corresponding equations

$$\frac{d^2 E_y}{dx^2} + n^2\left(x\right) E_y = \beta^2 E_y,$$

$$\frac{d^2 H_y}{dx^2} + n^2\left(x\right) H_y = \beta^2 H_y.$$

Both equations for the modes can be written in the more customary form

$$-\frac{d^2\psi}{dx^2}\left(k, x\right) + V\left(x\right)\psi\left(k, x\right) = k^2\psi\left(k, x\right). \qquad (1)$$

Here $V\left(x\right) = -n^2\left(x\right)$ is a piecewise constant function (constant in each of the layer), $k^2 = -\beta^2$ is the spectral parameter, and $\psi\left(x\right) = E_y\left(x\right)$ or $H_y\left(x\right)$.

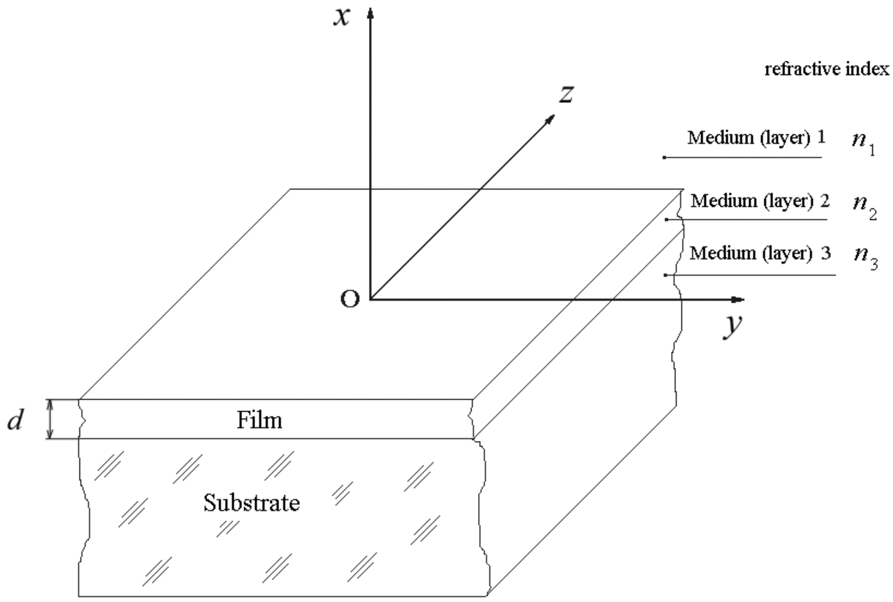

Figure 1: Waveguide is formed by media 1–3. The figure indications are: 1 is a framing medium or cover layer (air) with refractive index n_c; 2 is a waveguide layer (film) with a refractive index n_f; 3 is a substrate with refractive index n_s; d is the thickness of the waveguide layer. Film and substrate are homogeneous in the y and z directions, the substrate is usually much thicker than the film.

2. Formulation of the problem

Assumptions:

- A planar dielectric waveguide consists of homogeneous layers of isotropic materials, and the boundaries between the layer media are ideal and parallel to the xy plane.
- Electromagnetic radiation propagates in the longitudinal horizontal direction (along the z-axis) and is invariant along the transverse horizontal direction (along the y-axis).
- Electromagnetic radiation in the waveguide is monochromatic (harmonic time dependence).
- Electromagnetic radiation, for simplicity, is assumed to be linearly polarized.

The Maxwell equation in Cartesian coordinates associated with the waveguide geometry has the form (in the Gaussian units)

$$rot\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \quad rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{2}$$

and can be split into the two independent sets of linear ordinary differential equations

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = ik_0\varepsilon E_y, \quad H_x = -\frac{1}{ik_0\mu}\frac{\partial E_y}{\partial z}, \quad H_z = \frac{1}{ik_0\mu}\frac{\partial E_y}{\partial x}, \tag{3}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = ik_0\mu H_y, \quad E_x = \frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial z}, \quad E_z = -\frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial x}, \tag{4}$$

where $k_0 = \omega/c$ is the vacuum wave number, c is the speed of light in vacuum, ε and μ are, respectively, the dielectric constant and magnetic permeability, and $n^2 = \varepsilon\mu$ is the squared refractive index of a medium. In chosen coordinates, the boundary conditions

$$\vec{E}^\tau\Big|_1 = \vec{E}^\tau\Big|_2, \quad \vec{H}^\tau\Big|_1 = \vec{H}^\tau\Big|_2 \tag{5}$$

can be reduced to those for, respectively, TE and TM modes:

$$E_y|_1 = E_y|_2\,, \quad H_z|_1 = H_z|_2\,, \tag{6}$$

$$H_y|_1 = H_y|_2\,, \quad E_z|_1 = E_z|_2\,. \tag{7}$$

Solutions of the equations (3)–(6) and (4)–(7) yield vertical (along x axis) distributions of electromagnetic field for TE and TM modes, respectively. As functions of all the spacetime coordinates, electromagnetic fields of the modes can be written in the form

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}(x,y,z,t) = \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}(x)\exp\{i\omega t - ik_0\beta z\}\,. \tag{8}$$

Transforming (3) and (4) to the form

$$\frac{d^2E_y}{dx^2} + k_0^2\left(\varepsilon\mu - \beta^2\right)E_y(x) = 0, \quad H_z = \frac{1}{ik_0\mu}\frac{dE_y}{dx}, \quad H_x = -\frac{\beta}{\mu}E_y, \tag{9}$$

$$\varepsilon\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{dH_y}{dx}\right) + k_0^2\left(\varepsilon\mu - \beta^2\right)H_y(x) = 0, \quad E_z = -\frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial x}, \quad E_x = \frac{\beta}{\varepsilon}H_y, \tag{10}$$

both the sets can be written in the more customary form

$$-\frac{d^2\psi}{dx^2}(k,x) + V(x)\,\psi(k,x) = k^2\psi(k,x)\,. \tag{11}$$

Here $V(\tilde{x}) = -n^2(k_0x) = -\varepsilon(k_0x)\,\mu$ is a piecewise constant function (constant in each of the layer), $k^2 = -\beta^2$ is the spectral parameter, $\psi(\tilde{x}) = E_y(x)$ or $H_y(x)$, and $\tilde{x} = 2\pi(x/\lambda_0)$ is a dimensionless variable. Later on we use the notation x instead of the $\tilde{x}$.

The boundary conditions (6) and (7) hold for the function $\psi(\tilde{x})$ and its 'derivative'

$$\phi(\tilde{x}) = \frac{dE_y(x)}{dx} \quad \text{or} \quad \frac{1}{n^2(x)}\frac{dH_y(x)}{dx},$$

so that

$$\psi|_1 = \psi|_2\,, \quad \phi|_1 = \phi|_2\,. \tag{12}$$

The problem of finding waveguides modes is thus reduced to the problem (11)–(12) with a potential $V(x)$ for eigenvalues k and eigenfunction $\psi(k,x)$ obeying the asymptotic conditions (Figure 2)

$$V(x) \xrightarrow[x\to-\infty]{} V_-\,, \quad V(x) \xrightarrow[x\to\infty]{} V_+\,. \tag{13}$$

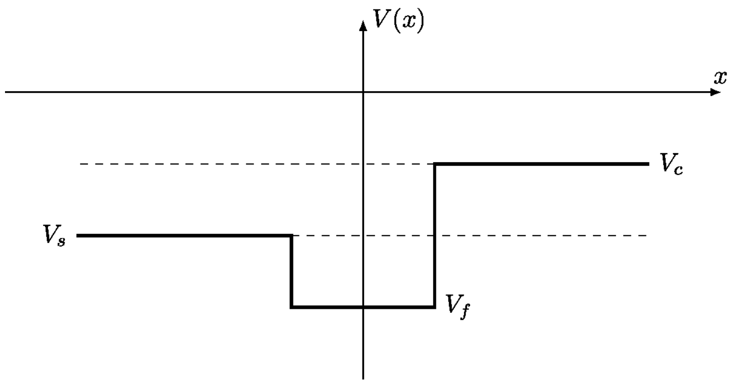

Figure 2: A schematic diagram of the potential.

In the problem (11)–(13), the operator spectrum consists of:

- a finite number of eigenvalues of the discrete spectrum $k_j = i\kappa_j \,:\, k_j^2 \in (\min V(x)\,,\ \min(V_-,\ V_+))$ and the corresponding eigenfunctions (guided modes);
- a continuous nondegenerate spectrum $k_- \,:\, k_-^2 \in (V_-,\ \infty)$ and the corresponding generalized eigenfunctions (substrate radiation modes);
- a continuous nondegenerate spectrum $k_+ \,:\, k_+^2 \in (V_+,\ \infty)$ and the corresponding generalized eigenfunctions (cover radiation modes).

In a multilayer waveguide with a piecewise constant potential $V(x)$, solutions to the problem (11)–(13) (in the notation of (9)–(10)) in the space of square integrable functions, that is, in the case of discrete spectrum $k_j = i\kappa_j$, were considered in a large number of research studies, both theoretical and computational. There are basic studies [9, 10, 11] and reviews [12, 13, 14, 15, 16] on integrated optics devoted to guided modes in waveguides. The pioneering [9, 10, 11] and recent works [17, 18, 19] on integrated optics are devoted to numerical methods of constructing the function $\psi_j(x)$ as a linear combination of a fundamental system of solutions of the equation (11) in each of the layers, with subsequent matching these functions at the layer interfaces according to (12).

3. Discrete and continuous eigenvalues and corresponding (classical and generalized) eigenfunctions

Schrödinger's operator on the x-axis,

$$\hat{H}y\left(x\right) \equiv -\frac{d^2y}{dx^2}\left(x\right) + V\left(x\right)y\left(x\right), \tag{14}$$

where the potential $V\left(x\right)$ obeying the conditions

$$\lim_{x\to-\infty} V\left(x\right) = V_-, \quad \lim_{x\to+\infty} V\left(x\right) = V_+, \tag{15}$$

$$\int_{-\infty}^{0} \left|V\left(x\right) - V_-\right|\left|x\right| dx < \infty, \quad \int_{0}^{\infty} \left|V\left(x\right) - V_+\right|\left|x\right| dx < \infty, \tag{16}$$

is essentially self-adjoint.

The equation

$$\hat{H}y = k^2 y \tag{17}$$

has a unique solution $y_-\left(k,x\right)$ having the asymptotic behaviour

$$\begin{aligned} \exp\left\{ip_-\left(k\right)x\right\}y_-\left(k,x\right) &\xrightarrow[x\to-\infty]{} 1, \\ \exp\left\{ip_-\left(k\right)x\right\}y'_-\left(k,x\right) &\xrightarrow[x\to-\infty]{} -ip_-\left(k\right), \end{aligned} \tag{18}$$

if $\operatorname{Im}\left(p_-\left(k\right)\right) \geqslant 0$, where $p_-\left(k\right)^2 \equiv k^2 - V_-$ with V_- obeying the condition (16). The equation

$$\hat{H}y = k^2 y \tag{19}$$

has a unique solution $y_+\left(k,x\right)$ having the asymptotic behaviour

$$\begin{aligned} \exp\left\{-ip_+\left(k\right)x\right\}y_+\left(k,x\right) &\xrightarrow[x\to\infty]{} 1, \\ \exp\left\{-ip_+\left(k\right)x\right\}y'_+\left(k,x\right) &\xrightarrow[x\to\infty]{} ip_+\left(k\right). \end{aligned} \tag{20}$$

if $\operatorname{Im}\left(p_+\left(k\right)\right) \geqslant 0$, where $p_+\left(k\right)^2 \equiv k^2 - V_+$ with V_+ obeying the condition (16). For real $p_-\left(k\right) \neq 0$, the pair of functions $\left\{y_-\left(k,x\right), \overline{y_-\left(k,x\right)}\right\}$ make up a fundamental system of solutions of the equation (8). Therefore, the solution $y_+\left(k,x\right)$ can be represented as

$$y_+\left(k,x\right) = a_-\left(k\right)\overline{y_-\left(k,x\right)} + b_-\left(k\right)y_-\left(k,x\right). \tag{21}$$

For real $p_+\left(k\right) \neq 0$, the pair of functions $\left\{y_+\left(k,x\right), \overline{y_+\left(k,x\right)}\right\}$ make up a fundamental system of solutions of the equation (8). Therefore, the solution $y_-\left(k,x\right)$ can be represented as

$$y_-\left(k,x\right) = a_+\left(k\right)\overline{y_+\left(k,x\right)} + b_+\left(k\right)y_+\left(k,x\right). \tag{22}$$

Coefficients $a_-(k)$ and $a_+(k)$ are inversely proportional to the transmission coefficients from the left to right and from the right to left, respectively; they can be expressed in terms of the Wronskian of $y_\pm(k,x)$, namely,

$$a_-(k) = W(y_-, y_+)/2ip_-(k) \quad \text{and} \quad a_+(k) = W(y_-, y_+)/2ip_+(k), \tag{23}$$

and can be analytically continued into the regions $\mathrm{Im}(p_-(k)) \geqslant 0$ and $\mathrm{Im}(p_+(k)) \geqslant 0$, respectively. Coefficients $b_-(k)$ and $b_+(k)$, which are proportional to the reflection coefficients from the left to right and from the right to left, respectively, can be expressed in terms of the Wronskian of $y_\pm(k,x)$ as

$$b_-(k) = W(y_+, \bar{y}_-)/2ip_-(k) \quad \text{and} \quad b_+(k) = W(\bar{y}_+, y_-)/2ip_+(k). \tag{24}$$

The zeros of the functions $a_-(k)$ and $a_+(k)$ are eigenvalues of the operator (14) with the normalized eigenfunctions

$$y_n(x) = c_n^- y_-(k_n, x) = c_n^+ y_+(k_n, x)\,. \tag{25}$$

In order to distinguish these solutions of the equation (19) from those constructed below and having other asymptotic behaviours, we introduce the new notations $V_s = V_-$ and $V_c = V_+$, $p_s(k) = p_-(k)$ and $p_c(k) = p_+(k)$.

In the region $\mathrm{Im} p_s(k)\,,\ k^2 > V_s$, (generalized) eigenfunctions of the operator (14) are

$$y_s(k,x) = \frac{1}{\sqrt{2\pi}}\left\{\overline{y_+(k,x)} + R_+(k)\,y_+(k,x)\right\}, \quad k^2 \in (V_s,\ \infty)\,. \tag{26}$$

In the region $\mathrm{Im} p_c(k)\,,\ k^2 > V_c$ (generalized) eigenfunctions of the operator (14) are

$$y_c(k,x) = \frac{1}{\sqrt{2\pi}}\left\{\overline{y_-(k,x)} + R_-(k)\,y_-(k,x)\right\}, \quad k^2 \in (V_c,\ \infty)\,, \tag{27}$$

where the reflection coefficients are given by

$$R_-(k) = \frac{b_-(k)}{a_-(k)} = \frac{W(y_+, \bar{y}_-)}{W(y_-, y_+)} \ \text{ and } R_+(k) = \frac{b_+(k)}{a_+(k)} = \frac{W(\bar{y}_+, y_-)}{W(y_-, y_+)}.$$

The corresponding expressions for the transmission coefficients are

$$T_-(k) = \frac{1}{a_-(k)} = \frac{2ip_-(k)}{W(y_-, y_+)} \quad \text{and} \quad R_+(k) = \frac{1}{a_+(k)} = \frac{2ip_+(k)}{W(y_-, y_+)}.$$

Theorem 1. *Let $y_\alpha(k,x)$ be a system of functions defined by the expressions (25)–(27). Then it makes up a complete orthonormal system of generalized eigenfunctions of the operator (14) so that the generalized Fourier transformation $f \to g$, given by*

$$f(x) = \int\limits_{V_s}^{\infty} g_s(k) y_s(k,x)\,dp_s(k) + \int\limits_{V_c}^{\infty} g_c(k) y_c(k,x)\,dp_c(k) + \sum_{n=1}^{N(\hat{H})} g_n(k) y_n(k,x)\,, \tag{28}$$

where

$$g_s(k) = \int\limits_{-\infty}^{\infty} f(x)\overline{y_s(k,x)}dx, \quad g_c(k) = \int\limits_{-\infty}^{\infty} f(x)\overline{y_c(k,x)}dx,$$
$$g_n(k) = \int\limits_{-\infty}^{\infty} f(x)\overline{y_n(k,x)}dx, \tag{29}$$

defines a unitary isomorphism between the equipped Hilbert spaces $S \subset L_2(\mathbb{R}) \subset S'$ *and* $S \subset L_2(M_k, dp(k)) \subset S'$*. Here* $L_2(M_k, dp(k))$ *is the Hilbert space of the square integrable functions on the space* M_k *endowed with the measure* $dp(k)$*, where* M_k *consists of the half-line* $\mathbb{R}_s^+ = \{k : k^2 > V_s\}$ *with the measure* $dp_s(k)$*, the half-line* $\mathbb{R}_c^+ = \{k : k^2 > V_c\}$ *with the measure* $dp_c(k)$*, and a finite collection of points* k_n*,* $n = 1, ..., N$ *with the point measures* $\delta(k - k_n)dk$*. The image of* $\hat{H}$ *under this isomorphism is the operator of multiplication by* k^2*.*

The introductory material of this section briefly reproduces the results of [8, 23].

4. Solution of the eigenvalue and eigenfunction problem

The guided modes in a multilayer waveguide are described by the problem (11)–(13) with values of the spectral parameter $k^2 \in (V_f, \min(V_s, V_c))$, so that $k^2 < 0$ (Figure 3). Therefore, it is convenient to introduce the notation $k_j = i\kappa_j \iff k_j^2 = -\kappa_j^2$, where k_j is an eigenvalue of the problem (11)—(13), and κ_j is the phase delay coefficient of the corresponding waveguide mode $\psi_j(x)$. We have $\psi_j(x) \in L_2(\mathbb{R})$, so that the asymptotic behaviour $\psi_j(x) \xrightarrow[x\to\pm\infty]{} 0$ takes place.

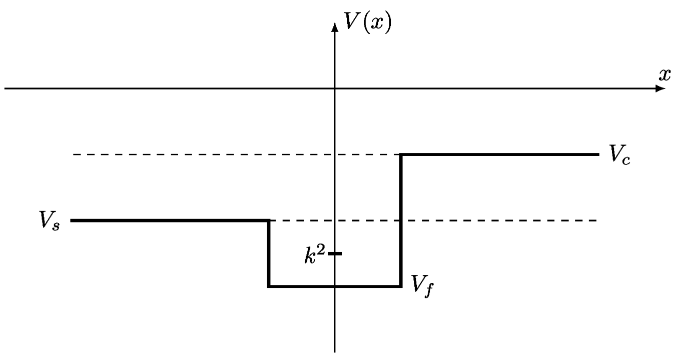

Figure 3: A schematic diagram, showing the location of the discrete spectral value relative to the potential, $k^2 \in (V_f, \min(V_s, V_c))$

By requiring that the functions $\psi^{TE} = E_y$, $\phi^{TE} = H_z = \frac{1}{ik_0\mu}\frac{dE_y}{dx} \sim \frac{d\psi^{TE}}{dx}$ satisfy the boundary conditions (12) at the points $x = a$ and $x = b$ (at the layer interfaces), we obtain a collection of particular solutions from the general solutions in the subregions $(-\infty, a)$, (a, b), (b, ∞); this collection gives the unique (up to a nonzero complex factor) solution of the problem (11)–(13). The analogous assertion also holds for the functions $\psi^{TM} = H_y$, $\phi^{TM} = E_z = -\frac{1}{ik_0\varepsilon}\frac{dH_y}{dx} \sim \frac{d\psi^{TM}}{dx}$.

So, in the region $(-\infty, a)$, the general solutions of the equation (11) (in which the coefficient V_s is supposed to be constant) obeying the asymptotic behaviour $\psi(x) \xrightarrow[x\to-\infty]{} 0$ have the forms (for TE and TM modes, respectively)

$$\psi_j^{TE}(x) = A_s \exp(\gamma_s x), \quad \phi_j^{TE}(x) = \frac{\gamma_s}{ik_0\mu_s} A_s \exp(\gamma_s x), \quad \gamma_s = \sqrt{V_s - k^2}, \quad \gamma_s > 0,$$

$$\psi_j^{TM}(x) = B_s \exp(\gamma_s x), \; \phi_j^{TM}(x) = -\frac{\gamma_s}{ik_0\varepsilon_s} B_s \exp(\gamma_s x), \; \gamma_s = \sqrt{V_s - k^2}, \quad \gamma_s > 0.$$

In the region (b, ∞), the general solutions of the equation (11) obeying the asymptotic behaviour $\psi(x) \xrightarrow[x\to\infty]{} 0$ have the forms

$$\psi_j^{TE}(x) = A_c \exp(-\gamma_c x), \; \phi_j^{TE}(x) = -\frac{\gamma_c}{ik_0\mu_c} A_c \exp(-\gamma_c x), \; \gamma_c = \sqrt{V_c - k^2}, \; \gamma_c > 0,$$

$$\psi_j^{TM}(x) = B_c \exp(-\gamma_c x), \; \phi_j^{TM}(x) = \frac{\gamma_c}{ik_0\varepsilon_c} B_c \exp(-\gamma_c x), \; \gamma_c = \sqrt{V_c - k^2}, \; \gamma_c > 0.$$

In the region (a, b), the general solutions of the equation (11) have the forms (for TE and TM modes, respectively)

$$\begin{gathered}
\psi_j^{TE}(x) = A_f^+ \exp(i\chi_f x) + A_f^- \exp(-i\chi_f x), \\
\phi_j^{TE}(x) = \frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp(i\chi_f x) - A_f^- \exp(-i\chi_f x)\right], \quad \chi_f = \sqrt{k^2 - V_f}, \\
\psi_j^{TM}(x) = B_f^+ \exp(i\chi_f x) + B_f^- \exp(-i\chi_f x), \\
\phi_j^{TM}(x) = -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp(i\chi_f x) - B_f^- \exp(-i\chi_f x)\right], \quad \chi_f = \sqrt{k^2 - V_f}.
\end{gathered}$$

Thus, these solutions are defined by (for TE and TM modes, respectively) the collection of the amplitudes $\left(A_s, A_f^+, A_f^-, A_c\right)^T$ obeying the system of linear equations

$$\psi_j^{TE}(a-0) = \psi_j^{TE}(a+0) \;\; \Leftrightarrow \;\; A_s \exp(\gamma_s a) \; = A_f^+ \exp(i\chi_f a) + A_f^- \exp(-i\chi_f a),$$

$$\begin{gathered}
\phi_j^{TE}(a-0) = \phi_j^{TE}(a+0) \;\; \Leftrightarrow \\
\Leftrightarrow \;\; \frac{\gamma_s}{ik_0\mu_s} A_s \exp(\gamma_s a) \; = \frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp(i\chi_f a) - A_f^- \exp(-i\chi_f a)\right],
\end{gathered}$$

$$\psi_j^{TE}(b-0) = \psi_j^{TE}(b+0) \;\; \Leftrightarrow \;\; A_f^+ \exp(i\chi_f b) + A_f^- \exp(-i\chi_f b) = A_c \exp(-\gamma_c b),$$

$$\begin{gathered}
\phi_j^{TE}(b-0) = \phi_j^{TE}(b+0) \;\; \Leftrightarrow \\
\Leftrightarrow \;\; \frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp(i\chi_f b) - A_f^- \exp(-i\chi_f b)\right] \; = -\frac{\gamma_c}{ik_0\mu_c} A_c \exp(-\gamma_c b),
\end{gathered}$$

and the collection of the amplitudes $\left(B_s, B_f^+, B_f^-, B_c\right)^T$ obeying the system of linear equations

$$\psi_j^{TM}(a-0) = \psi_j^{TM}(a+0) \;\;\Leftrightarrow\;\; B_s \exp(\gamma_s a) \;= B_f^+ \exp(i\chi_f a) + B_f^- \exp(-i\chi_f a)\,,$$

$$\begin{aligned} &\phi_j^{TM}(a-0) = \phi_j^{TM}(a+0) \;\;\Leftrightarrow \\ &\qquad \Leftrightarrow\;\; -\frac{\gamma_s}{ik_0\varepsilon_s} B_s \exp(\gamma_s a) \;= -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp(i\chi_f a) - B_f^- \exp(-i\chi_f a)\right], \end{aligned}$$

$$\psi_j^{TM}(b-0) = \psi_j^{TM}(b+0) \;\;\Leftrightarrow\;\; B_f^+ \exp(i\chi_f b) + B_f^- \exp(-i\chi_f b) = B_c \exp(-\gamma_c b)\,,$$

$$\begin{aligned} &\phi_j^{TM}(b-0) = \phi_j^{TM}(b+0) \;\;\Leftrightarrow \\ &\qquad \Leftrightarrow\;\; -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp(i\chi_f b) - B_f^- \exp(-i\chi_f b)\right] \;= \frac{\gamma_c}{ik_0\varepsilon_c} B_c \exp(-\gamma_c b)\,. \end{aligned}$$

These systems have the form of homogeneous ones,

$$\hat{M}^{TE}(k)\left(A_s, A_f^+, A_f^-, A_c\right)^T = \vec{0}, \quad \hat{M}^{TM}(k)\left(B_s, B_f^+, B_f^-, B_c\right)^T = \vec{0},$$

therefore they have nontrivial solutions if and only if the solvability conditions (for TE and TM modes, respectively),

$$\det \hat{M}^{TE}(k) = 0, \quad \det \hat{M}^{TM}(k) = 0,$$

are hold. Solving these equations then gives the desired discrete values k_j^{TE}, k_j^{TM} : $k_j^2 \in (V_f,\, \min(V_s,\, V_c))$.

This method of computing the eigenvalues k_j^{TE} and k_j^{TM}, and the corresponding eigenvectors $\left(A_s, A_f^+, A_f^-, A_c\right)^T$ and $\left(B_s, B_f^+, B_f^-, B_c\right)^T$ is described in the works [9, 10] and in the monographs [12, 13, 14, 15, 16]. A detailed consideration of the method is given in the JINR preprint [17] where one also find computer realizations of both the dispersion relations (Figure 4) and the corresponding distributions along the x-axis of electric and magnetic field strengthes for TE and TM modes (Figure. 5).

5. Computing the cover radiation modes

We consider solutions of the problem (11)–(13) in the region of continuous spectrum, namely, $k^2 \in (V_c,\, \infty)$ (Figure 6).

Solutions $y_+(k,x)$ and $y_-(k,x)$ of the problem (19) satisfy the Jost asymptotic conditions [22, 23]. In contrast, the solutions $y_s(k,x)$ and $y_c(k,x)$ of the problem (19) satisfy the asymptotic conditions for the scattering by the potential $V(x)$ [24]. In particular, the asymptotic behavior of the solutions $y_c(k,x)$ corresponds to the scattering problem for a plane wave running on the potential $V(x)$ from the right

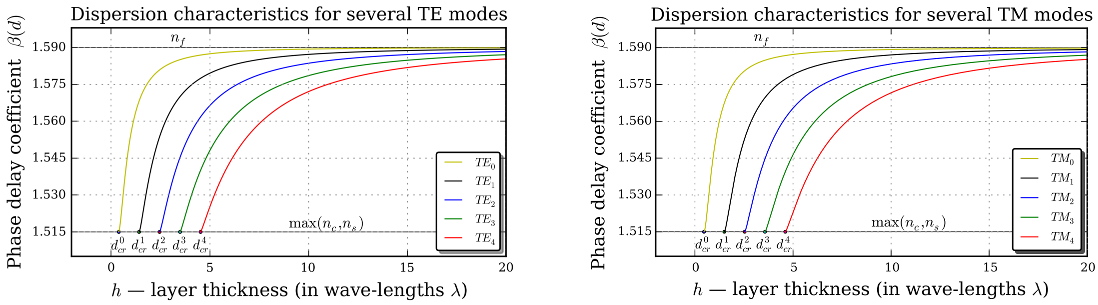

Figure 4: The dispersion curves corresponding to the first spectral values: phase delay coefficient vs layer thickness

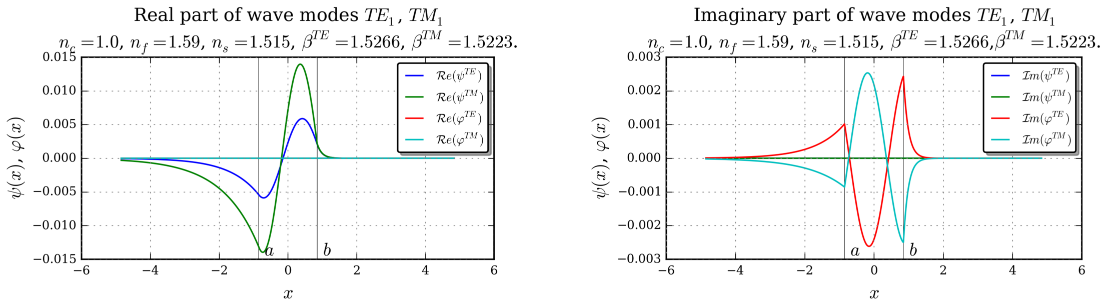

Figure 5: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the guided modes

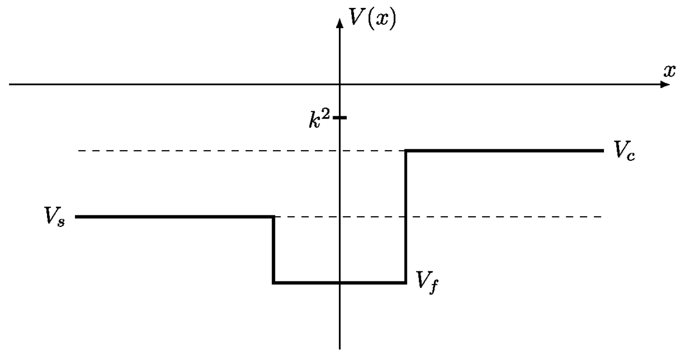

Figure 6: A schematic diagram, showing the location of the cover mode's continuous spectral value relative to the potential.

(from $+\infty$), which then turns out to be partially reflected backward with the reflection coefficient $R_-(k)$ and partially transmitted over the potential (in the form of a plane wave propagating to $-\infty$) with the transmission coefficient $T_-(k)$. All the solutions $y_c(k, x)$ for all $k^2 \in (V_c, \infty)$ have the same asymptotic behaviour.

Next, the asymptotic behavior of solutions $y_s(k, x)$ corresponds to the scattering problem for a plane wave running on the potential $V(x)$ from the left (from $-\infty$) to right, which is partially reflected to the left with the reflection coefficient $R_+(k)$ and partially transmitted over the potential to the right, either as a plane wave propagating with the transmission coefficient $T_+(k)$ or as an evanescent wave decaying with the weight $A_c(k)$ for, respectively, $k^2 \in (V_c, \infty)$ and $k^2 \in (V_s, V_c)$.

The cover radiating modes are considered in the works [7, 8, 22, 23] as generalized eigenfunctions for the equations (11) with the boundary conditions (12) and the spectral values $k \in (V_c, \infty)$. As in the case of the guided modes, they can be constructed by matching the general solutions of the equations (11) at the boundaries of the regions $(-\infty, a)$, (a, b), and (b, ∞). Solving of the one-dimensional potential scattering problem with coincident asymptotic behaviours is given in the works [26, 27].

So, in the region $(-\infty, a)$, the general solutions of the equation (11) (in which the coefficient V_s is supposed to be constant) for TE and TM modes have, respectively, the forms

$$\psi_c^{TE}(k,x) = T_-^{TE}(k)\exp(-ip_s x)\,,\ \phi_c^{TE}(x) = -\frac{p_s}{k_0\mu_s}T_-^{TE}(k)\exp(-ip_s x)\,,$$

and

$$\psi_c^{TM}(k,x) = T_-^{TM}(k)\exp(-ip_s x)\,,\ \phi_c^{TM}(x) = \frac{p_s}{k_0\varepsilon_s}T_-^{TM}(k)\exp(-ip_s x)\,.$$

In the region (b, ∞), the general solutions of the equation (11) have the forms

$$\begin{aligned}
\psi_c^{TE}(k,x) &= \exp(-ip_c x) + R_-^{TE}(k)\exp(ip_c x)\,,\\
\phi_c^{TE}(x) &= -\frac{p_c}{k_0\mu_c}\left[\exp(-ip_c x) - R_-^{TE}(k)\exp(ip_c x)\right],\\
\psi_c^{TM}(k,x) &= \exp(-ip_c x) + R_-^{TM}(k)\exp(ip_c x)\,,\\
\phi_c^{TM}(x) &= \frac{p_c}{k_0\varepsilon_c}\left[\exp(-ip_c x) - R_-^{TM}(k)\exp(ip_c x)\right].
\end{aligned}$$

In the region (a, b), the general solutions of the equation (11) have the forms (for TE and TM modes, respectively)

$$\begin{aligned}
\psi_c^{TE}(x) &= A_f^+\exp(i\chi_f x) + A_f^-\exp(-i\chi_f x)\,,\\
\phi_c^{TE}(x) &= \frac{\chi_f}{k_0\mu_f}\left[A_f^+\exp(i\chi_f x) - A_f^-\exp(-i\chi_f x)\right],\\
\psi_c^{TM}(x) &= B_f^+\exp(i\chi_f x) + B_f^-\exp(-i\chi_f x)\,,\\
\phi_c^{TM}(x) &= -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+\exp(i\chi_f x) - B_f^-\exp(-i\chi_f x)\right].
\end{aligned}$$

Thus, these solutions are defined by (for TE and TM modes, respectively) the collection of the amplitudes $\left(T_-^{TE}, A_f^+, A_f^-, R_-^{TE}\right)^T$ obeying the system of linear equations

$$T_-^{TE}(k)\exp(-ip_s a) = A_f^+\exp(i\chi_f a) + A_f^-\exp(-i\chi_f a)\,,$$

$$-\frac{p_s}{k_0\mu_s}T_-^{TE}(k)\exp(-ip_s a) = \frac{\chi_f}{k_0\mu_f}\left[A_f^+\exp(i\chi_f a) - A_f^-\exp(-i\chi_f a)\right],$$

$$A_f^+\exp(i\chi_f b) + A_f^-\exp(-i\chi_f b) = \exp(-ip_c b) + R_-^{TE}(k)\exp(ip_c b),$$

$$\frac{\chi_f}{k_0\mu_f}\left[A_f^+\exp(i\chi_f b) - A_f^-\exp(-i\chi_f b)\right] = -\frac{p_c}{k_0\mu_c}\left[\exp(-ip_c b) - R_-^{TE}(k)\exp(ip_c b)\right],$$

and the collection of the amplitudes $\left(T_-^{TM}, A_f^+, A_f^-, R_-^{TM}\right)^T$ obeying the system of linear equations

$$T_-^{TM}(k)\exp(-ip_s a) = B_f^+\exp(i\chi_f a) + B_f^-\exp(-i\chi_f a),$$

$$\frac{p_s}{k_0\varepsilon_s}T_-^{TM}(k)\exp(-ip_s a) = -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+\exp(i\chi_f a) - B_f^-\exp(-i\chi_f a)\right],$$

$$B_f^+\exp(i\chi_f b) + B_f^-\exp(-i\chi_f b) = \exp(-ip_c b) + R_-^{TM}(k)\exp(ip_c b),$$

$$-\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+\exp(i\chi_f b) - B_f^-\exp(-i\chi_f b)\right] = \frac{p_c}{k_0\varepsilon_c}\left[\exp(-ip_c b) - R_-^{TM}(k)\exp(ip_c b)\right].$$

In both cases we obtain inhomogeneous systems of algebraic linear equations of the forms

$$\hat{M}^{TE}(k)\left(T_-^{TE}, A_f^+, A_f^-, R_-^{TE}\right)^T = \left(0, 0, \exp(-ip_c b), -\frac{p_c}{k_0\mu_c}\exp(-ip_c b)\right)^T,$$

$$\hat{M}^{TE}(k)\left(T_-^{TM}, B_f^+, B_f^-, R_-^{TM}\right)^T = \left(0, 0, \exp(-ip_c b), \frac{p_c}{k_0\varepsilon_c}\exp(-ip_c b)\right)^T,$$

so that the solutions exist for any $k^2 \in (V_c, \infty)$ and are unique up to a nonzero complex factor (Figure 7).

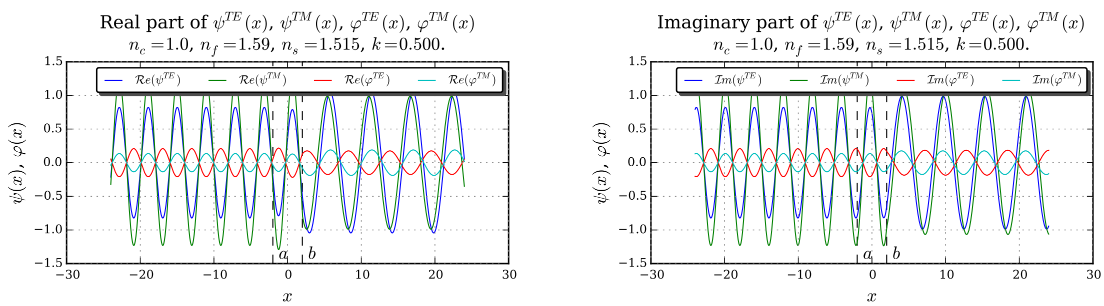

Figure 7: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the cover radiation modes

6. Computing the substrate radiation modes

The substrate radiation modes are considered in the works [7, 8, 22, 23] as generalized eigenfunctions for the equations (11) with the boundary conditions (12). Solution of the one-dimensional potential scattering problem with coincident asymptotic behaviours is given in the works [20, 21, 25, 26]. The solutions have different forms for the two regions of the spectral parameter values, namely, $k^2 \in (V_s, V_c)$ and $k^2 \in (V_c, \infty)$. However, as in the case of the guided modes, the whole solution can be constructed by matching the general solutions of the equations (11) at the boundaries of the regions $(-\infty, a)$, (a, b), and (b, ∞).

A) The spectral value location for the case $k^2 \in (V_s, V_c)$ is shown on the schematic diagram in Figure 8.

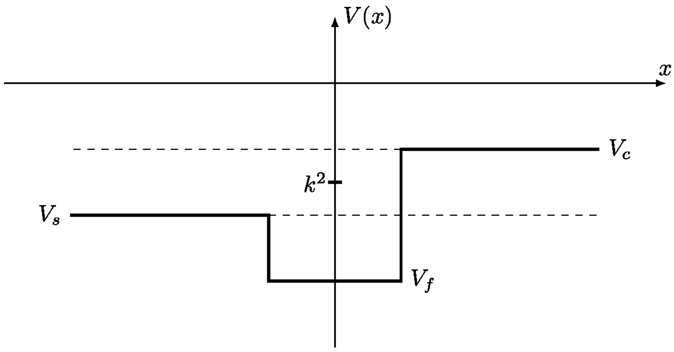

Figure 8: A schematic diagram, showing the location of the substrate mode's continuous spectral value relative to the potential, $k^2 \in (V_s, V_c)$.

In the region $(-\infty, a)$, the general solutions of the equation (11) for $k^2 \in (V_s, V_c)$ are

$$\psi_s^{TE}(k,x) = \exp\left(ip_s(k)\,x\right) + R_+^{TE}(k)\exp\left(-ip_s(k)\,x\right),$$
$$\phi_s^{TE}(k,x) = \frac{p_s}{k_0\mu_s}\left[\exp\left(ip_s(k)\,x\right) - R_+^{TE}(k)\exp\left(-ip_s(k)\,x\right)\right],$$
$$\psi_s^{TM}(k,x) = \exp\left(ip_s(k)\,x\right) + R_+^{TM}(k)\exp\left(-ip_s(k)\,x\right),$$
$$\phi_s^{TM}(k,x) = -\frac{p_s}{k_0\varepsilon_s}\left[\exp\left(ip_s(k)\,x\right) - R_+^{TM}(k)\exp\left(-ip_s(k)\,x\right)\right].$$

In the region (a, b), the general solutions of the equation (11) for$k^2 \in (V_s, V_c)$ are

$$\psi_s^{TE}(k,x) = A_f^+\exp\left(i\chi_f x\right) + A_f^-\exp\left(-i\chi_f x\right),$$
$$\phi_f^{TE}(k,x) = \frac{\chi_f}{k_0\mu_f}\left[A_f^+\exp\left(i\chi_f x\right) - A_f^-\exp\left(-i\chi_f x\right)\right],$$
$$\psi_s^{TM}(k,x) = B_f^+\exp\left(i\chi_f x\right) + B_f^-\exp\left(-i\chi_f x\right),$$
$$\phi_f^{TM}(k,x) = -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+\exp\left(i\chi_f x\right) - B_f^-\exp\left(-i\chi_f x\right)\right].$$

In the region (b, ∞), the general solutions of the equation (11) for $k^2 \in (V_s, V_c)$ have, owing to the asymptotic vanishing, the forms

$$\psi_s^{TE}(k,x) = A_c \exp(-\gamma_c x), \quad \phi_s^{TE}(x) = -\frac{\gamma_c}{ik_0\mu_c} A_c \exp(-\gamma_c x),$$

$$\psi_s^{TM}(k,x) = B_c \exp(-\gamma_c x), \quad \phi_s^{TM}(x) = \frac{\gamma_c}{ik_0\varepsilon_c} B_c \exp(-\gamma_c x).$$

Thus, these solutions are defined by (for TE and TM modes, respectively) the collection of the amplitudes $\left(R_+^{TE}, A_f^+, A_f^-, A_c\right)^T$ obeying the system of linear equations

$$\exp(ip_s(k)a) + R_+^{TE}(k)\exp(-ip_s(k)a) = A_f^+ \exp(i\chi_f a) + A_f^- \exp(-i\chi_f a),$$

$$\frac{p_s}{k_0\mu_s}\left[\exp(ip_s(k)a) - R_+^{TE}(k)\exp(-ip_s(k)a)\right] = \\ = \frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp(i\chi_f a) - A_f^- \exp(-i\chi_f a)\right],$$

$$A_f^+ \exp(i\chi_f b) + A_f^- \exp(-i\chi_f b) = A_c \exp(-\gamma_c b),$$

$$\frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp(i\chi_f b) - A_f^- \exp(-i\chi_f b)\right] = -\frac{\gamma_c}{ik_0\mu_c} A_c \exp(-\gamma_c b),$$

and the collection of the amplitudes $\left(R_+^{TM}, B_f^+, B_f^-, B_c\right)^T$ obeying the system of linear equations

$$\exp(ip_s(k)a) + R_+^{TM}(k)\exp(-ip_s(k)a) = B_f^+ \exp(i\chi_f a) + B_f^- \exp(-i\chi_f a),$$

$$-\frac{p_s}{k_0\varepsilon_s}\left[\exp(ip_s(k)a) - R_+^{TM}(k)\exp(-ip_s(k)a)\right] = \\ = -\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp(i\chi_f a) - B_f^- \exp(-i\chi_f a)\right],$$

$$B_f^+ \exp(i\chi_f b) + B_f^- \exp(-i\chi_f b) = B_c \exp(-\gamma_c b),$$

$$-\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp(i\chi_f b) - B_f^- \exp(-i\chi_f b)\right] = \frac{\gamma_c}{ik_0\varepsilon_c} B_c \exp(-\gamma_c b).$$

In both cases we obtain inhomogeneous systems of algebraic linear equations of the forms

$$\hat{M}^{TE}(k)\left(R_+^{TE}, A_f^+, A_f^-, A_c\right)^T = \left(-\exp(ip_s a), -\frac{p_s}{k_0\mu_s}\exp(ip_s a), 0, 0\right)^T,$$

$$\hat{M}^{TM}(k)\left(R_+^{TM}, B_f^+, B_f^-, B_c\right)^T = \left(-\exp(ip_s a), \frac{p_s}{k_0\varepsilon_s}\exp(ip_s a), 0, 0\right)^T,$$

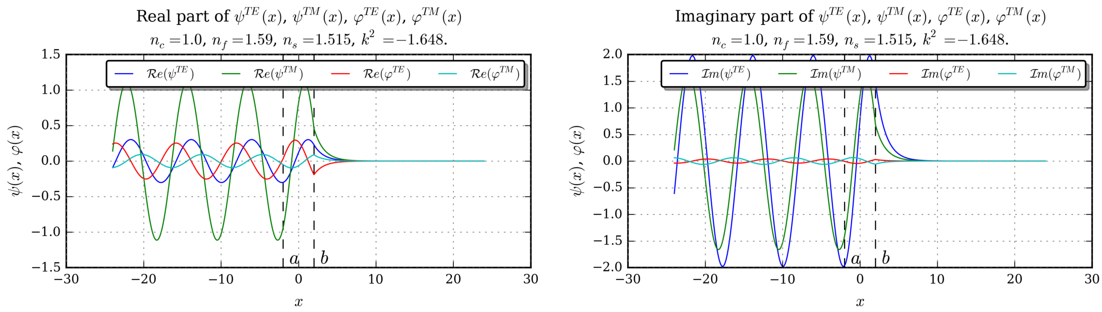

Figure 9: The curves for the field strength (along the vertical axis) for the substrate radiation modes decaying in the cover layer.

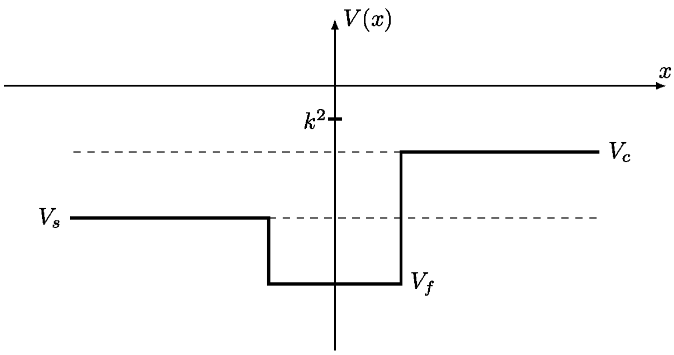

Figure 10: A schematic diagram, showing the location of the substrate mode's continuous spectral value relative to the potential, $k^2 \in (V_c, \infty)$.

so that the solutions exist for any $k^2 \in (V_c, \infty)$ and are unique up to a nonzero complex factor (Figure 9).

B) The spectral value location for the case $k^2 \in (V_c, \infty)$ is shown on the schematic diagram in Figure 10.

The guided modes in a multilayer waveguide are described by the problem (11)–(13) with values of the spectral parameter $k^2 \in (V_f, \min(V_s, V_c))$, so that $k^2 < 0$ (Figure 3).

In the regions $(-\infty, a)$ and (a, b), the general solutions with $k^2 \in (V_c, \infty)$ have the same form as those for $k^2 \in (V_s V_c)$, while in the region (b, ∞) they are of the forms

$$\psi_s^{TE}(k,x) = T_+^{TE}(k) \exp(ip_c(k)x), \quad \phi_s^{TE}(k,x) = \frac{p_c(k)}{k_0 \mu_c} T_+^{TE}(k) \exp(ip_c(k)x),$$

$$\psi_s^{TM}(k,x) = T_+^{TM}(k) \exp(ip_c(k)x), \quad \phi_s^{TM}(k,x) = -\frac{p_c(k)}{k_0 \varepsilon_c} T_+^{TM}(k) \exp(ip_c(k)x).$$

Consequently, the second pair of the matching equations for TE and TM modes takes, respectively, the form

$$A_f^+ \exp\left(i\chi_f b\right) + A_f^- \exp\left(-i\chi_f b\right) = T_+^{TE}\left(k\right) \exp\left(ip_c\left(k\right) b\right),$$

$$\frac{\chi_f}{k_0\mu_f}\left[A_f^+ \exp\left(i\chi_f b\right) - A_f^- \exp\left(-i\chi_f b\right)\right] = \frac{p_c\left(k\right)}{k_0\mu_c} T_+^{TE}\left(k\right) \exp\left(ip_c\left(k\right) b\right)$$

and

$$B_f^+ \exp\left(i\chi_f b\right) + B_f^- \exp\left(-i\chi_f b\right) = T_+^{TM}\left(k\right) \exp\left(ip_c\left(k\right) b\right),$$

$$-\frac{\chi_f}{k_0\varepsilon_f}\left[B_f^+ \exp\left(i\chi_f b\right) - B_f^- \exp\left(-i\chi_f b\right)\right] = -\frac{p_c\left(k\right)}{k_0\varepsilon_c} T_+^{TM}\left(k\right) \exp\left(ip_c\left(k\right) b\right).$$

As above, in both cases we obtain inhomogeneous systems of algebraic linear equations of the forms

$$\hat{M}^{TE}\left(k\right)\left(R_+^{TE}, A_f^+, A_f^-, T_+^{TE}\right)^T = \left(-\exp\left(ip_s a\right), -\frac{p_s}{k_0\mu_s}\exp\left(ip_s a\right), 0, 0\right)^T$$

and

$$\hat{M}^{TM}\left(k\right)\left(R_+^{TM}, B_f^+, B_f^-, T_+^{TM}\right)^T = \left(-\exp\left(ip_s a\right), \frac{p_s}{k_0\varepsilon_s}\exp\left(ip_s a\right), 0, 0\right)^T,$$

so that the solutions exist for any $k^2 \in (V_c, \infty)$ and are unique up to a nonzero complex factor (Figure 11).

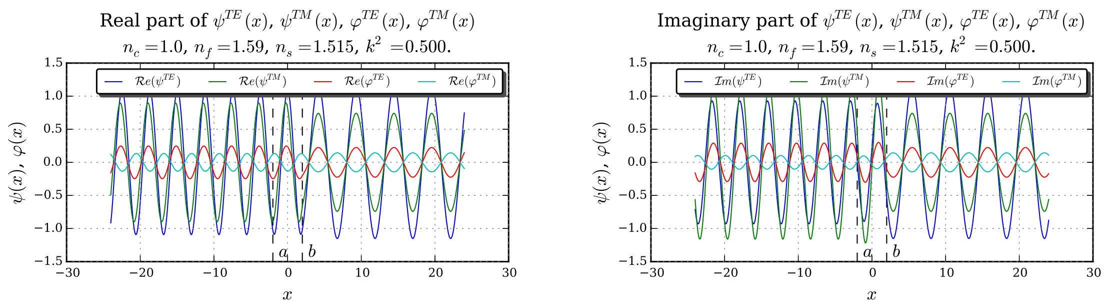

Figure 11: The curves for the field strength (along the vertical axis) for the substrate radiation modes oscillating in the cover layer.

7. Conclusion

Solution of many problems of integrated optics includes spectral analysis and spectral synthesis based on the complete system of solutions of a second-order differential operator governing the waveguide modes in an open waveguide. In the simplest case, a regular waveguide operator is essentially self-adjoint and has the mixed spectrum: the finite nondegenerate discrete spectrum and two branches of the continuous spectrum. This complete set of modes is used to describe the waveguide propagation of electromagnetic radiation by the comparison-of-waveguides method, and also can be used in the incomplete Galerkin method for integrated optical waveguides.

In this work we consider the special, but the most widespread case of a multilayer waveguide. We give numerical solutions for the three types of waveguide modes: guided modes, substrate radiation modes, and cover radiation modes.

The main theoretical result, on which our analysis in this article is based, is the following theorem.

Theorem 2. *Let $y_\alpha(k,x)$ be a system of functions defined by the expressions (25)–(27). Then it makes up a complete orthonormal system of generalized eigenfunctions of operator (14) so that generalized Fourier transformation defines a unitary isomorphism between the equipped Hilbert spaces $S \subset L_2(\mathbb{R}) \subset S'$ and $S \subset L_2(M_k, dp(k)) \subset S'$. Here $L_2(M_k, dp(k))$ is the Hilbert space of the square integrable functions on the space M_k endowed with the measure $dp(k)$, where M_k consists of the half-line $\mathbb{R}_s^+ = \{k : k^2 > V_s\}$ with the measure $dp_s(k)$, the half-line $\mathbb{R}_c^+ = \{k : k^2 > V_c\}$ with the measure $dp_c(k)$, and a finite collection of points k_n, $n = 1, ..., N$ with the point measures $\delta(k - k_n)\, dk$. The image of $\hat{H}$ under this isomorphism is the operator operator of multiplication by k^2.*

For the eigenvalue problem with a piecewise constant potential (in a multilayer waveguide) in the space of square integrable functions, numerical solutions for discrete spectrum $k_j = i\kappa_j$ were considered in a large number of research studies, both theoretical and computational. There are basic studies [9, 10, 11] and reviews [12, 13, 14, 15, 16] on integrated optics devoted to guided modes in waveguides. The pioneering [9, 10, 11] and recent works [17, 18, 19] on integrated optics are devoted to numerical methods of constructing the function $\psi_j(x)$ as a linear combination of a fundamental system of solutions of the equation (11) in each of the layers, with subsequent matching these functions at the layer interfaces according to (12).

In this article we present numerical results for the cover radiation modes and substrate radiation modes. Our modelling method consists in reducing the initial potential scattering problem (in the case of the continuous spectrum) to the equivalent ones for the Jost functions: the Jost solution from the left for the substrate radiation modes and the Jost solution from the right for the cover radiation modes.

Acknowledgements. The authors are grateful to prof. Tsirulev A. N. for his considerable assistance.

References

[1] Yariv A. *Coupled-Mode Theory for Guided-Wave Optics.* IEEE Quant. Electronis 1973, **9**, No 9, pp. 919–933

[2] Haus H.A., Huang W.P., Kawakami S., and Whiteker N.A. *Coupled-Mode Theory of Optical Waveguides.* IEEE J. Lightwave Technolgy 1987, **5**, No 1, pp. 16–23

[3] Katsenelenbaum B.Z. *Theory of Irregular Waveguideswith Slowly Varying Parameters.* 1961, Acad. Sci. USSR, Moscow (in Russian])

[4] Shevchenko V.V. *Continuous Transitions in Open Waveguides.* 1971, Golem, Boulder, Colorado

[5] Sveshnikov A.G. *The basis for a method of calculating irregular waveguides.* Computational Mathematics and Mathematical Physics 1963, **3**, No 1, pp. 170–179

[6] Sveshnikov A.G. *A substantiation of a method for computing the propagation of electromagnetic oscillations in irregular waveguides.* Computational Mathematics and Mathematical Physics 1963, **3**, No 2, pp. 314–326

[7] Shevchenko V.V. *On the spectral decomposition in eigenfunctions and associated functions of a nonself-adjoint Sturm-Liouville problem on the whole axis.* Differential Equations 1979, **15**, pp. 2004–2020 (in Russian)

[8] Sevastianov L.A. *The complete system of modes of an open planar waveguide.* In Proceedings of the Conference on Lasers in Science, Engineering, and Medicine, 1995, IRE RAN, Moscow, pp. 72-76

[9] Deryugin L.N., Marchuk A.N., and Sotin V.E. *Properties of planar asymmetrical dielectric waveguides on a substrate of dielectric.* Izv. Vyssh. Uchebn. Zaved., Ser. Radioelektron. 1967, **10**, No 2, pp. 134–141 (in Russian)

[10] Zolotov E.M., Kiselev V.A., and Sychugov V.A. *Optical phenomena in thin-film waveguides.* Soviet Physics-Uspekhi 1974, **112**, issue 2, pp. 231-273

[11] Goncharenko A.M., Deryugin L.N., Prohorov A.M., and Schipulo G.P. *On the development of integrated optics in the USSR.* Zh. Prikl. Spectroscop. 1978, **XXIX**, No 6, pp. 987–997 (in Russian)

[12] Marcuze D. *Theory of Dielectric Optical Waveguides.* 1974, New York, Academic Press

[13] *Introduction to Integrated Optics.* 1974, Ed. by M. Barnoski, New York, Plenum Press

[14] *Integrated Optics.* 1975, Ed. by Tamir T., Heidelberg, Springer-Verlag; Tamir T. *Guided-Wave Optoelectronics.* 1990, Springer-Verlag, Berlin

[15] Adams M.J. *An Introduction to Optical Waveguides.* 1981, New York, Wiley

[16] Snyder A.W. and Love J. *Optical Waveguide Theory.* 1983, London, Chapman and Hall

[17] Ayrjan E.A., Egorov A.A., Michuk E.N., Sevastyanov A.L., Sevastianov L.A., and Stavtsev A.V. *Representations of guided modes of integrated-optical multilayer thin-film waveguides.* Preprint JINR E11-2011-31, Dubna, 2011, 52 pp.

[18] Song D. and Lu Y.Y. *Pseudospectral Modal Method for Computing Optical Wave guide Modes.* IEEE J. Lightwave Technolgy 2014, **32**, issue 8, pp. 1612–1630

[19] Huang Ch-Ch., Huang Ch-Ch., and Yang J-Y. *An Efficient Metod for Computing Optical Waveguides With Discontinuous Refractive Index Profile Using Spectral Collocation Method With Domain Decomposition.* IEEE J. Lightwave Technolgy 2003, **21**, No 10. pp. 2284–2296

[20] Faddeev L.D. *Properties of the S-matrix of the one-dimensional Schrödinger equation.* Am. Math. Soc. Transl. (Ser. 2) 1967, **65**, pp. 139–166

[21] Buslayev V.S. and Fomin V.M. *An inverse scattering problem for the one-dimensional Schrödinger equation on the entire axis.* Vestnik Leningrad. Univ. 1962, **17**, pp. 56–64 (in Russian)

[22] Deift P. and Trubowitz D. *Inverse scattering on the line.* Comm. Pure Appl. Math. 1979, **32**, pp. 121–251

[23] Cohen A. and Kappeler T. *Scattering and inverse scattering for steplike potentials in the Schrödinger equation.* Indiana Univ. Math. J. 1985, **34**, pp. 127–180

[24] Berezin F.A. and Shubin M.A. *The Schroedinger equation.* 1991, Kluwer Academic Publishers

[25] Boyd J.P. *Chebyshev and Fourier Spectral Methods.* Second edition, 2001, Dover, New York

[26] Boyd J.P. *A Chebyshev/radiation function pseudospectral method for wave scattering.* Computers in Physics, 1981, **4**, pp. 83–85

[27] Clemente P.C.M., de Oliveira H.P., and Rodrigues E.L. *Black Hole Scattering via Spectral Methods.* Braz. J. Phys. 2014, **44**, pp. 128–137