http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 2, No 3, p. 11 – 27 (2014)

Integration of highly oscillatory functions

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Received 24 September 2014, in final form 10 October. Published 15 October 2014.

Abstract. Ability to calculate integrals of rapidly oscillating functions is crucial for solving many problems in optics, electrodynamics, quantum mechanics, nuclear physics, and many other areas. The article considers the method of computing oscillatory integrals with the help of the transition to the numerical solution of the system of ordinary differential equations without boundary conditions. Using the differentiation matrices reduces the problem to solving a system of linear algebraic equations.
We have proposed several variants of constructing differentiation matrices, leading to effective and sustainable methods of solving the systems of linear equations with the subsequent calculation of integrals of rapidly oscillating functions for a wide class of nonlinear and even non monotonic phase functions. We have also proposed a simple option for computing integrals of rapidly oscillating functions in the case of linear phase functions that allows us to express this is almost analytically. The advantages of the proposed methods are demonstrated by a number of numerical examples.

Keywords: integration of rapidly oscillating functions, Filon method, Levin method, Chebyshev differential matrix, ill-conditioned matrices, sustainable methods for solving systems of linear algebraic equations
MSC numbers: 65D32; 41A55; 60D05

The authors thank partial support RFBR Grants Nos. 13-01-00595 and 14-01-00628.

1. Introduction

The calculation of integrals of rapidly oscillating functions is one of the most difficult, but very important procedures in solving applied problems in optics, electrodynamics, quantum mechanics, and many other areas of science and technology. The traditional algorithms for computing the integral of rapidly oscillating functions, which are used in widely distributed computing packages such as Maple, Mathcad, and others work fairly well for a not very large number of the simplest cases. For example, in the design of real lighting systems the results of such calculations are unreliable. There is a need to develop reliable and accurate numerical methods and computational algorithms for the integration of a wide class of rapidly oscillating functions.

To evaluate the computational complexity of this procedure, consider the integral of the form:

$$I = \int_a^b f(x)e^{i\omega g(x)}dx \equiv \int_a^b F(x)dx, \tag{1}$$

where the region of integration is such that the constant of oscillation $\omega \gg 1$ is a "large" quantity; the amplitude $f(x)$ and the phase $g(x)$ are sufficiently smooth and not oscillating functions. Even in the case of a linear function $g(x) \equiv x$ the real and imaginary parts $Re\left(f(x)e^{i\omega x}\right)$ and $Im\left(f(x)e^{i\omega x}\right)$ have at $[a, b]$ about $\omega(b-a)/\pi$ zeros, because the phase factor is in the range of integration of the $\omega(b-a)$ order. The number of periods $F(x)$ is proportional to $\omega(b-a)/(2\pi)$ and each period has 2 roots. Thus, the region of integration has about $\omega(b-a)/\pi$ zeros. This means that, for example, the quantity of a uniform step $h = (b-a)/n \ll 1/\omega$ in the numerical integration needs to be small and/or must be large $n \gg \omega(b-a) > \omega(b-a)/\pi$. Thus, at each period of the function $F(x)$ it is necessary to choose both a large number of grid nodes, and a high degree polynomial for approximation, which is disadvantageous from any point of view — all this leads to large amounts of computation [1].

For the integrals of functions with linear phase the Filon method [2] is often used, which works reliably and accurately. It is based on the construction of composite quadrature formulas, in which the interpolation polynomial of low power for the phase $f(x)$ is used at each partial interval, and further integration is performed accurately. For example, the computation of the expression

$$\int_a^b x^k e^{i\omega x}dx$$

may be carried out by integrating by parts or by using the relation

$$\int_a^b x^k e^{i\omega x}dx = \frac{1}{(-i\omega)^{k+1}}\left[\Gamma(1+k, -i\omega a) - \Gamma(1+k, -i\omega b)\right],$$

where Γ is the incomplete gamma function.

In applications, however, irregular oscillations leading to integrals of the form

$$\int_a^b x^k e^{i\omega g(x)} dx$$

are often found. In this case all depends on the type of the oscillator $g(x)$. In the case of a simple form of the oscillator the Filon's method can be used, for example, for polynomial functions of the type$g(x) = x^r$. But even for functions $g(x) = x^3 - x$ or $g(x) = \cos x$ the Filon's method is not applicable [1].

The Levin's collocation method [4] is suitable for finding the oscillatory integrals with more complex phase function. It consists in the transition to solving a system of ordinary differential equations for obtaining the primitive of the integrand that is defined by a function $p(x)$ satisfying the condition

$$\frac{d}{dx}\left[p(x)e^{i\omega g(x)}\right] = f(x)e^{i\omega g(x)}. \tag{2}$$

After obtaining the function $p(x)$, one can calculate the value of the integral of the oscillating function

$$I[f] = \int_a^b f e^{i\omega g} dx = \int_a^b \frac{d}{dx}\left[p e^{i\omega g}\right] dx = p(b)e^{i\omega g(b)} - p(a)e^{i\omega g(a)}. \tag{3}$$

We can rewrite the condition (2) in the equivalent form $p' + i\omega g' p = f$, or, for brevity, $L[p] = f$ where

$$L[p] = p' + i\omega g' p. \tag{4}$$

Note that the method does not use the boundary conditions, as any particular solution gives a solution to the problem of the value of the definite integral. In this method, the problem of computing the integral is replaced by the "equivalent" problem of finding the values of the primitive function at two points at the ends of the integration interval $[a, b]$, allowing to calculate the value of the integral $I[f]$ with the formula (3).

Consider in this context the problem of finding a primitive of the integrand, namely the function $p(x)$ satisfying (2). The spectral methods for obtaining the required function are to represent it in the form of the expansion

$$p(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \tag{5}$$

on the basis of $\{\phi_k(x)\}_1^\infty$ in some Hilbert space. To achieve acceptable accuracy it's necessary to use a sufficiently large number $(n + 1)$ of members of the series.

Consider the “operator” $L(p)$: $L[p](x) = f(x)$. It is required that with some coefficients c_k, $k = 1, ..., n$ the solution should satisfy

$$L\left[\sum_{k=0}^{n} c_k \phi_k(x)\right](x) = f(x). \quad (6)$$

The collocation method that is related to the spectral methods consists in finding the coefficients of the expansion of the solution of the differential equation (4) by the values $L[p](x) = f(x)$ in a given set of collocation nodes $\{x_0, ..., x_n\}$, i.e. the coefficients c_k can be determined as the solution of the system of collocation:

$$L[p](x_0) = f(x_0), ..., L[p](x_n) = f(x_n). \quad (7)$$

In determining the approximate value of the integral $I[f]$ in the form

$$Q^L[f] = \int_a^b L(p)e^{i\omega g}dx = \int_a^b \frac{d}{dx}\left[pe^{i\omega g}\right]dx = p(b)e^{i\omega g(b)} - p(a)e^{i\omega g(a)} \quad (8)$$

the following estimate of the approximation error [5] is valid:

$$I[f] - Q^L[f] = O(\omega^{-1}),$$

— in the case where the boundary points are not included in the number of the grid nodes;

$$I[f] - Q^L[f] = O(\omega^{-2}),$$

— in the case when the boundary points are included in the number of the grid nodes.

These estimates imply a very simple practical conclusion: the presence of the boundary points within the grid points increases the accuracy (an order of magnitude more) of the solution.

Thus, the problem of calculating the rapidly oscillating integral (1) can be reduced to solving a system of linear algebraic equations (SLAE) (7). With a suitable choice of the points of the approximation, their location within the interval of the integration and their number, the increase of accuracy of the solution can be achieved [6, 7]. The conditionality matrix of the system may also depend on it (6).

2. Approximation to the required function using the Chebyshev polynomials

The Chebyshev polynomials of first kind have the best use in practical calculations, among various basis systems of polynomials for approximation on finite segments. Therefore we examine the Chebyshev polynomials of first kind as basic functions. Let us assume that we know the values of n-th degree polynomial in $(n+1)$ points

— $x_1, ..., x_n$. Then these values determine the polynomial in the only way, so its derivatives' values in these points - $p'(x) = dp(x)/dx$. The value of the derivative in every point can be expressed as a linear combination of these values in these points. This relation can be expressed in the matrix form:

$$\begin{pmatrix} p'(x_0) \\ \vdots \\ p'(x_n) \end{pmatrix} = \begin{pmatrix} d_{00} & \cdots & d_{0n} \\ \vdots & \ddots & \vdots \\ d_{n0} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{pmatrix} \tag{9}$$

Thematrix $D = \{d_{jk}\}$ is called the differentiation matrix (in Gauss–Lobatto nodes).

If you choose the Chebyshev polynomials of the first kind as basic functions, and points of arrays are Gauss–Lobatto nodes, then the elements of the matrix of differentiation can be found using the following formulas [8]:

$$\begin{aligned}
d_{jk} &= \frac{(-1)^{k-j}}{x_j - x_k}, \quad 0 < j \neq k < n, \\
d_{kk} &= -\frac{1}{2}\frac{x_k}{1 - x_k^2}, \quad 0 < k < n, \\
d_{00} = \frac{1}{6}(1 + 2n^2), &\quad d_{nn} = -\frac{1}{6}(1 + 2n^2), \\
d_{0k} &= 2\frac{(-1)^k}{1 - x_k}, \quad 0 < k < n, \\
d_{k0} &= -\frac{1}{2}\frac{(-1)^k}{1 - x_k}, \quad 0 < k < n, \\
d_{kn} &= \frac{1}{2}\frac{(-1)^{n-k}}{1 + x_k}, \quad 0 < k < n, \\
d_{nk} &= -2\frac{(-1)^{n-k}}{1 + x_k}, \quad 0 < k < n, \\
d_{0n} = \frac{1}{2}(-1)^n, &\quad d_{n0} = -\frac{1}{2}(-1)^n.
\end{aligned}$$

Such matrix is called the Chebyshev Differentiation Matrix.

Note: a easy to check that the sum of column of the Chebyshev Matrix equals zero vector, so the Matrix is degenerate [8] (see also [9, 10]).

As above we are going to find the solution in the form of an infinite expansion in the orthogonal basis functions $\tilde{p}(x) = \sum_{k=0}^{\infty} a_k \phi_k(x)$ and approximate it with the finite series of this sum. We assume that the Chebyshev polynomials of first kind are chosen as basic functions to find the derivatives of the required solution. The expression for the derivative is:

$$\frac{d}{dx}\left(\sum_{k=0}^{n} a_k T_k(x)\right) = \sum_{k=0}^{n} b_k T_k(x). \tag{10}$$

The recurrent relation $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \quad k = 2, 3, ...$ with the starting conditions $T_0(x) = 1, \quad T_1(x) = x$ allows us to develop the recurrent relation between the coefficients of expansion of the original function and the coefficients of its derivative:

$$\begin{bmatrix} 1 & 0 & -1/2 & & & & \\ & 1/4 & 0 & -1/4 & & & \\ & & 1/6 & 0 & -1/6 & & \\ & & & 1/8 & 0 & -1/8 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \frac{1}{2n-2} & 0 \\ & & & & & & \frac{1}{2n} \end{bmatrix} \times \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \tag{11}$$

We can derive the formula from this recurrent relation that connects these coefficients in reverse order with the use of the differentiation matrix A in the dimension of coefficients:

$$\mathbf{Aa} = \mathbf{b} \tag{12}$$

where $A_{ij} = \begin{cases} (1/c_i) \times 2j, & \text{if} \quad i > j \quad \text{and} \quad \text{sum} \quad i + j \ - \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$

Here $0 \leqslant i, j \leqslant n, \quad c_i = \begin{cases} 2, & i = 0, \\ 1 & i > 0. \end{cases}$

For example, with $n = 4$ the transformation is

$$\mathbf{Aa} = \mathbf{b} \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 3 & 0 \\ & 0 & 4 & 0 & 8 \\ & & 0 & 6 & 0 \\ & & & 0 & 8 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{13}$$

The (7) equation in this model of solution is converted to:

$$\frac{2}{b-a}\mathbf{TAa} + i \ \cdot \ diag(g')\mathbf{Ta} = f, \tag{14}$$

where T is the array of $(n + 1) \times (n + 1)$ size, each line of it consists of the values of $(n + 1)$th Chebyshev polynomial $T_0(x), ..., T_n(x)$, calculated in the xth node of random array.

Let's add some details to this formula:

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \dots & T_{n,0} \\ T_{0,1} & T_{1,1} & T_{2,1} & \dots & T_{n,1} \\ T_{0,2} & T_{1,2} & T_{2,2} & \dots & T_{n,2} \\ \dots & \dots & \dots & \dots & \dots \\ T_{0,n} & T_{1,n} & T_{2,n} & \dots & T_{n,n} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 & ... \\ & 0 & 4 & 0 & ... \\ & & 0 & 6 & ... \\ & & & ... & ... \\ & & & & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} +$$

$$+ i \begin{bmatrix} g'_0 & & & & \\ & g'_1 & & & \\ & & g'_2 & & \\ & & & ... & \\ & & & & g'_n \end{bmatrix} \begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \dots & T_{n,0} \\ T_{0,1} & T_{1,1} & T_{2,1} & \dots & T_{n,1} \\ T_{0,2} & T_{1,2} & T_{2,2} & \dots & T_{n,2} \\ \dots & \dots & \dots & \dots & \dots \\ T_{0,n} & T_{1,n} & T_{2,n} & \dots & T_{n,n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ ... \\ f_n \end{bmatrix}. \quad (15)$$

Here we use $T_i,\ j = T_i(x_j)$ and $g'_j = g'(x_j)$ notations to make the derivation shorter.

The solution of this system of linear algebraic equations relative to $a = (a_0, a_1, ..., a_n)$ coefficients of the solution expansion in the basic functions allows us to define the approximate value of the integral with the use of Eq. (3).

3. The method of quadratures based on application of the Chebyshev differentiation matrix in physical space

If we consider the integral on a segment $x \in [a, b]$, then for transferring to the standard region of definition of the Chebyshev polynomials $[-1,\ 1]$ it is possible to change the variables $x = \frac{b-a}{2}t + \frac{b+a}{2}$, $t \in [-1,\ 1]$. Then

$$p'(x) = \frac{2}{b-a}p'(t). \quad (16)$$

According to the introduced linear transformation the Gauss–Lobatto nodes $t_j = \cos(\frac{\pi_j}{N-1})$ in the initial co-ordinates are of the form

$$x_j = \frac{b-a}{2} \cos\left(\frac{\pi_j}{N-1}\right) + \frac{b+a}{2}, \quad j = 0, 1, ..., N-1.$$

The vectors of function values and their derivatives in the Gauss–Lobatto nodes are calculated according to the formulas

$$\begin{aligned} \vec{p} &= [p(x_0), p(x_1), ..., p(x_n)]^T, \\ \vec{p}\,' &= [p'(x_0), p'(x_1), ..., p'(x_n)]^T, \\ \vec{g}\,' &= [g'(x_0), g'(x_1), ..., g'(x_n)]^T, \\ f &= [f(x_0), f(x_1), ..., f(x_n)]^T. \end{aligned} \quad (17)$$

It is obvious that according to the definition of the Chebyshev differentiation matrix, we can express $\mathbf{p}'$ in the vector-matrix form from (16) with the account of (9),

$$\mathbf{p}' = \frac{2}{b-a}\mathbf{D}\mathbf{p}. \tag{18}$$

In the grid nodes of x_j, the equations (7) will be expressed of a form

$$\begin{bmatrix} p'(x_0) \\ p'(x_1) \\ \vdots \\ p'(x_{N-1}) \end{bmatrix} + i \begin{bmatrix} g'(x_0)p(x_0) \\ g'(x_1)p(x_1) \\ \vdots \\ g'(x_{N-1})p(x_{N-1}) \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{bmatrix}. \tag{19}$$

Let us substitute (17) and (18) into (19). Then the system can be expressed in the form:

$$\frac{2}{b-a}\mathbf{D}\mathbf{p} + i \cdot diag(g')\mathbf{p} = f, \tag{20}$$

or

$$(D + i\Lambda)p = \lambda f, \tag{21}$$

where $\lambda = (b-a)/2$, $\Lambda = diag(\lambda g'(x_0), \lambda g'(x_1), \ldots, \lambda g'(x_{N-1}))$ is a diagonal matrix. The solution of the system (21) contains $p(b)$ and $p(a)$ and the required integral is calculated according to the formula (8). Thus, the integral calculation is reduced to solving a system of linear equations (21).

The differentiation matrix D is singular, but its condition number improves with addition of the diagonal matrix $i\Lambda$ which has no zeros on the diagonal [11]. We will assume that in one or several points of the Gauss–Lobatto grid the function g' becomes zero. Then it is possible to choose a new grid with the different quantity of nodes, so that g' does not become zero on it. Moreover, in all those cases when g' has a finite number of zeros on an investigated segment, for some sufficiently great number of grid nodes there exists the Gaussa-Lobatto grid with N points, not containing any zero of the function g'. Besides, the application of the Tikhonov regularization method also allows to provide the steady solution of the system and, hence, calculation of the required integral.

4. Method of quadratures based on application of Chebyshev differentiation matrix in the momentum space.

A rather simple system of equations can be derived in the case where the function $g(x)$ is linear, i.e. $g'(x) = const = g'$. Then, pre-multiplying the first and the last equation of system (15) by $1/\sqrt{2}$ multiplying the system to the left by the transposed matrix $\mathbf{T}^T$ and using the properties of the discrete orthogonality of the Chebyshev polynomials defined on the Gauss–Lobatto grid, we can bring the system (15) to the upper triangular form

$$\begin{bmatrix} \sum\limits_{i=0}^{n} T_0^2(x_i) & 0 & 0 & \dots & 0 \\ 0 & \sum\limits_{i=0}^{n} T_0^2(x_i) & 0 & \dots & 0 \\ 0 & 0 & \sum\limits_{i=0}^{n} T_0^2(x_i) & \dots & 0 \\ \dots & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & \dots & \sum\limits_{i=0}^{n} T_0^2(x_i) \end{bmatrix} \times$$

$$\times \left(\begin{bmatrix} 0 & 1 & 0 & 3 & \dots \\ & 0 & 4 & 0 & \dots \\ & & 0 & 6 & \dots \\ & & & \dots & \dots \\ & & & & 0 \end{bmatrix} + i \begin{bmatrix} g' & & & & \\ & g' & & & \\ & & g' & & \\ & & & \dots & \\ & & & & g' \end{bmatrix} \right) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} =$$

$$= \begin{bmatrix} T_0(x_0)/\sqrt{2} & T_0(x_1) & T_0(x_2) & \dots & T_0(x_n)/\sqrt{2} \\ T_1(x_0)/\sqrt{2} & T_1(x_1) & T_1(x_2) & \dots & T_1(x_n)/\sqrt{2} \\ T_2(x_0)/\sqrt{2} & T_2(x_1) & T_2(x_2) & \dots & T_2(x_n)/\sqrt{2} \\ \dots & \dots & \dots & \dots & \dots \\ T_n(x_0)/\sqrt{2} & T_n(x_1) & T_n(x_2) & \dots & T_n(x_n)/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_0/\sqrt{2} \\ f_1 \\ f_2 \\ \dots \\ f_n/\sqrt{2} \end{bmatrix} \quad (22)$$

Here

$$\sum_{i=0}^{n} T_j^2(x_i) = \frac{1}{2}T_j^2(x_0) + T_j^2(x_1) + T_j^2(x_2) + \ldots + T_j^2(n_{n-1}) + \frac{1}{2}T_j^2(x_n), \quad j = 0, \ldots, n.$$

Given the specific values of the norms of the Chebyshev polynomials on the Gauss-Lobatto grid the system (15) is reduced to the form

$$\begin{bmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n/2 & 0 & \dots & 0 \\ 0 & 0 & n/2 & \dots & 0 \\ \dots & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & \dots & n \end{bmatrix} \times$$

$$\times \left(\begin{bmatrix} 0 & 1 & 0 & 3 & \dots \\ & 0 & 4 & 0 & \dots \\ & & 0 & 6 & \dots \\ & & & \dots & \dots \\ & & & & 0 \end{bmatrix} + i \begin{bmatrix} g' & & & & \\ & g' & & & \\ & & g' & & \\ & & & \dots & \\ & & & & g' \end{bmatrix} \right) \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} =$$

$$= \begin{bmatrix} T_0(x_0)/\sqrt{2} & T_0(x_1) & T_0(x_2) & \dots & T_0(x_n)/\sqrt{2} \\ T_1(x_0)/\sqrt{2} & T_1(x_1) & T_1(x_2) & \dots & T_1(x_n)/\sqrt{2} \\ T_2(x_0)/\sqrt{2} & T_2(x_1) & T_2(x_2) & \dots & T_2(x_n)/\sqrt{2} \\ \dots & \dots & \dots & \dots & \dots \\ T_n(x_0)/\sqrt{2} & T_n(x_1) & T_n(x_2) & \dots & T_n(x_n)/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_0/\sqrt{2} \\ f_1 \\ f_2 \\ \dots \\ f_n/\sqrt{2} \end{bmatrix} \quad (23)$$

Let us consider a more general case, where $g'(x) \neq const$, but $g'(x) \neq 0$ for all $x_j = \cos\left(\frac{\pi j}{N-1}\right)$, then the resulting system of linear algebraic equations with a triangular matrix is still not degenerate. In both cases it is easy to get the solution of the system with triangular nonsingular matrix, for example, by the reverse sweep of the Gauss method

$$\begin{bmatrix} ing' & n & 0 & 3n & \dots & 0 \\ 0 & ing'/2 & 2n & 0 & \dots & 0 \\ 0 & 0 & ing'/2 & 3n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & ing' \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} =$$

$$= \begin{bmatrix} T_0(x_0)/\sqrt{2} & T_0(x_1) & T_0(x_2) & \dots & T_0(x_n)/\sqrt{2} \\ T_1(x_0)/\sqrt{2} & T_1(x_1) & T_1(x_2) & \dots & T_1(x_n)/\sqrt{2} \\ T_2(x_0)/\sqrt{2} & T_2(x_1) & T_2(x_2) & \dots & T_2(x_n)/\sqrt{2} \\ \dots & \dots & \dots & \dots & \dots \\ T_n(x_0)/\sqrt{2} & T_n(x_1) & T_n(x_2) & \dots & T_n(x_n)/\sqrt{2} \end{bmatrix} \begin{bmatrix} f_0/\sqrt{2} \\ f_1 \\ f_2 \\ \dots \\ f_n/\sqrt{2} \end{bmatrix} \quad (24)$$

This system can be expressed as

$$\begin{bmatrix} ing' & n & 0 & 3n & \dots & 0 \\ 0 & ing'/2 & 2n & 0 & \dots & 0 \\ 0 & 0 & ing'/2 & 3n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & ing' \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{n} T_0(x_k) f(x_k) \\ \sum\limits_{k=0}^{n} T_1(x_k) f(x_k) \\ \sum\limits_{k=0}^{n} T_2(x_k) f(x_k) \\ \dots \\ \sum\limits_{k=0}^{n} T_n(x_k) f(x_k) \end{bmatrix}, \quad (25)$$

wherein $\sum\limits_{j=0}^{n}$, $j = 0, \dots, n$ has the same meaning as in formula (23), dividing both sides of the system by n we obtain

$$\begin{bmatrix} ig' & 1 & 0 & 3 & \dots & 0 \\ 0 & ig'/2 & 2 & 0 & \dots & 0 \\ 0 & 0 & ig'/2 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & ig' \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \sum\limits_{k=0}^{n} T_0(x_k) f(x_k) \\ \frac{1}{n} \sum\limits_{k=0}^{n} T_1(x_k) f(x_k) \\ \frac{1}{n} \sum\limits_{k=0}^{n} T_2(x_k) f(x_k) \\ \dots \\ \frac{1}{n} \sum\limits_{k=0}^{n} T_n(x_k) f(x_k) \end{bmatrix}. \quad (26)$$

The system of equations (26) in the case of $g'(x) = const$ admits the analytical solution. With the help of the integral (3) it can also be expressed in the analytical form.

5. Alternative method of representation of the Chebyshev differentiation matrix, based on the use of other recurrence relations

Using the recurrence relations [8],

$$xT_n(x) = \frac{1}{2}\left(T_{n+1}(x) + T_{|n-1|}(x)\right), \quad n = 0, 1, ..., \tag{27}$$

$$\frac{d}{dx}T_n(x) = \frac{n}{2}\frac{T_{n-1}(x) - T_{n+1}(x)}{1 - x^2}, \quad |x| \neq 1, \tag{28}$$

we obtain a different from (13) expression for the Chebyshev polynomial derivative of n-order

$$\frac{d}{dx}T_n(x) = \frac{n}{2}\frac{T_{n-1}(x) - T_{n+1}(x)}{1 - x^2} = \frac{n}{2}\frac{T_{n-1}(x) - \left[2xT_n(x) - T_{|n-1|}(x)\right]}{1 - x^2} = \\ = n\frac{T_{|n-1|}(x) - xT_n(x)}{1 - x^2}. \tag{29}$$

The usage of this expression for the presentation of the derivative of the desired solution when $|x| \neq 1,\ x \in (0, 1)$ leads to the equation

$$\sum_{j=0}^{n} \frac{jT_{|j-1|}(x)}{1 - x^2}a_j - \sum_{j=0}^{n} \frac{jxT_j(x)}{1 - x^2}a_j + iq'(x)\sum_{j=0}^{n} T_j(x)a_j = f(x). \tag{30}$$

To express the differential equation in boundary points a well-known [8] expression for Chebyshev polynomial derivatives can be used at points$x = \pm 1$: $T_n'(\pm 1) = \pm n^2 T_n(\pm 1), \quad n = 0, 1, 2, ...$ Then we obtain that on the interval boundary $[-1, 1]$ differential equation is satisfied in case when

$$\sum_{j=0}^{n} (\pm 1)^j n^2 T_j(\pm 1)a_j + iq'(x)\sum_{j=0}^{n} T_j(\pm 1)a_j = f(\pm 1). \tag{31}$$

Taking into consideration, that $T_n(\pm 1) = (\pm 1)^n$, let us write out algebraic equation set (29)–(30) for the Gauss-Lobatto grids in formula indication (15)

$$\left(\begin{bmatrix} 0 & 1 & 4 & 9 & \dots & n^2 \\ 0 & \frac{T_{0,1}-x_1T_{1,1}}{1-x_1^2} & 2\frac{T_{1,1}-x_1T_{2,1}}{1-x_1^2} & 3\frac{T_{2,1}-x_1T_{3,1}}{1-x_1^2} & \dots & n\frac{T_{n-1,1}-x_1T_{n,1}}{1-x_1^2} \\ 0 & \frac{T_{0,2}-x_2T_{1,2}}{1-x_2^2} & 2\frac{T_{1,2}-x_2T_{2,2}}{1-x_2^2} & 3\frac{T_{2,2}-x_2T_{3,2}}{1-x_2^2} & \dots & n\frac{T_{n-1,2}-x_2T_{n,2}}{1-x_2^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{T_{0,n-1}-x_{n-1}T_{1,n-1}}{1-x_{n-1}^2} & 2\frac{T_{1,n-1}-x_{n-1}T_{2,n-1}}{1-x_{n-1}^2} & 3\frac{T_{2,n-1}-x_{n-1}T_{3,n-1}}{1-x_{n-1}^2} & \dots & n\frac{T_{n-1,n-1}-x_{n-1}T_{n,n-1}}{1-x_{n-1}^2} \\ 0 & 1 & -4 & 9 & \dots & (-1)^{n-1}n^2 \end{bmatrix} + \right.$$

$$\left. +i\begin{bmatrix} q_0' & & & \vdots & \\ & q_1' & & \vdots & \\ & & q_2' & \vdots & \\ \dots & \dots & \dots & \ddots & \dots \\ & & & \vdots & q_n' \end{bmatrix}\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \dots & T_{n,0} \\ T_{0,1} & T_{1,1} & T_{2,1} & \dots & T_{n,1} \\ T_{0,2} & T_{1,2} & T_{2,2} & \dots & T_{n,2} \\ \dots & \dots & \dots & \dots & \dots \\ T_{0,n} & T_{1,n} & T_{2,n} & \dots & T_{n,n} \end{bmatrix}\right) \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \dots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{n-1} \\ f_n \end{bmatrix}. \quad (32)$$

As before, in case, when $g'(x) \neq 0$, the system (31) is non-degenerate and can be solved by one of the standard methods. The solution of this system regarding the expansion coefficient of the solution in the Chebyshev polynomials makes it easy to calculate the values of $p(a)$ and $p(b)$:

$$p(a) = \sum_{j=0}^{n} a_j, \quad p(b) = \sum_{j=0}^{n} (-1)^j a_j. \quad (33)$$

The value of the required integral is calculated using the formula (8). Thus, this differentiation matrix representation allows to display clearly non-degeneracy of the system of linear equations (31) in the cases, when $g'(x) \neq 0$ in all the grid points. Previously, it was shown that for the phase function with a finite number of zeros of its derivative, such Gauss-Lobatto node grid can be chosen to solve the resulting system of linear equations in a sustainable manner.

6. Numerical examples

To demonstrate the results of our research, we have chosen such numerical examples, which are widespread (as example 1), or which are of fundamental importance (as examples 2.3).

Example 1. For solving practical problems it is often necessary to calculate the Bessel functions with high indices for large values of the argument. There are a large number of procedures for the calculation of the Bessel functions [12]. We compare our proposed approach with the BesselJN procedure from the well-known package Cephes Math Library [13], for testing the functionality of our method. We use one of the well-known representations of the Bessel function in the form of integral from a rapidly oscillating function

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\tau - x\sin\tau)}d\tau. \quad (34)$$

Figure 1 shows a graph of the integrand function.

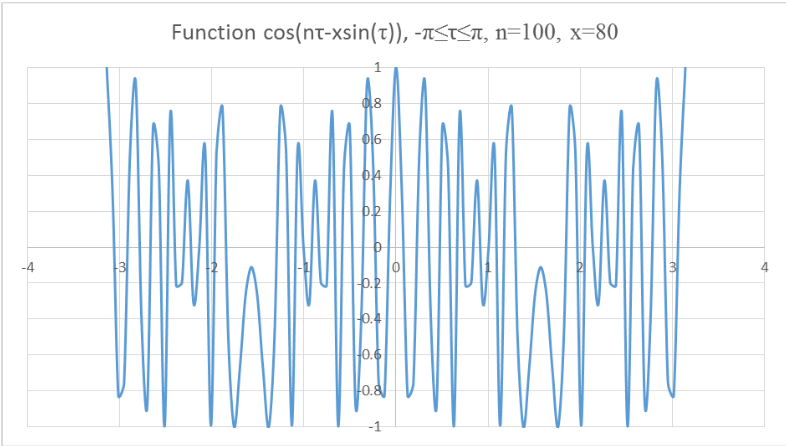

Figure 1: The graph of the real part of the integrand function in the integral representation of the Bessel function (33) 100-th order at the point $x = 80$

The Bessel function was calculated at different values of the indices and the large values of the arguments (see Fig. 2). The function values calculated by this algorithm were compared with the values defined in the program BesselJN from a well-known package [13] by Stephen L. Moshier. The deviation (in the graph, multiplied by 10^{11} — red line) for all calculated values is less than 2.62e-11, for the calculations with double precision.

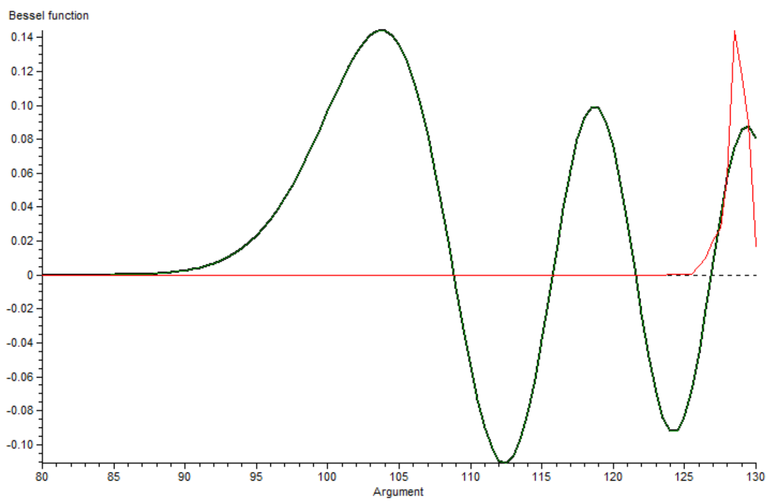

Figure 2: Comparison of the calculated values of the Bessel function of the 100-th order on the interval [80, 130], received according to the BesselJN program and our method

The solution of the following tasks analyzed in works [4, 11, 14] requires the selection of special parameters (system matrix norm), when using our algorithm the selection of the specific parameters isn't required, the decision turns out steady anyway. We compare the results with the exact decision, listed in the same works.

Example 2. In Ref. [11] as the first example it was proposed to calculate the integral $I = \int_0^1 \sin(x)e^{i500(x_x^2)}dx$, proposed originally in work [4]. The exact value of this integral was given in the same work

$$I = (4.59859397840143 - i \times 3.15443542737400) \times 10^{-4}.$$

We have carried out the integral calculation on the Gauss–Lobatto nodes with the number of points from 20 to 34 by methods described by us. We solved the obtained system of linear equations by the LU-method of decomposition or SVD-method [3]. In all cases the identical results were received with computer accuracy (see Fig. 3). Numerical values of the relative error are plotted in the Figure in comparison with the exact value depending on the number of points of a grid.

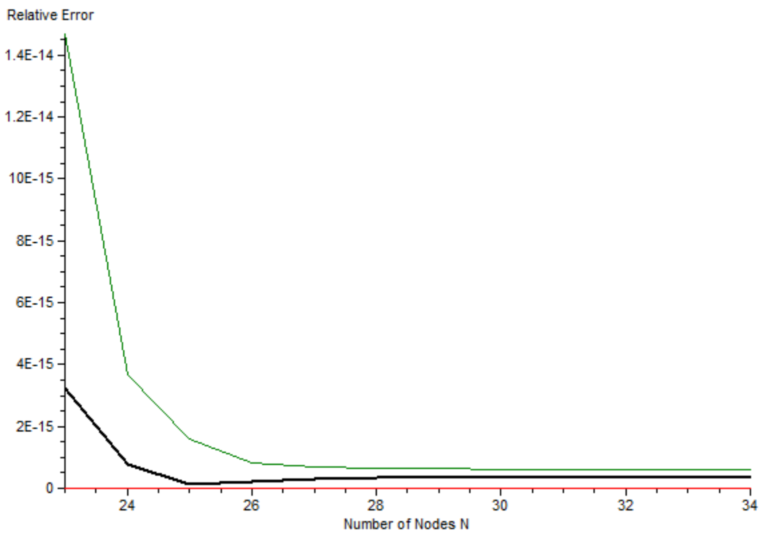

Figure 3: Relative error of computation depending of number of nodes: green line – imaginary part (minimum equals 6.2397625708279e-16 at N= 32), black (minimum equals 1.4829300966583e-16 at N= 25) – real part

Further expansion of a number of collocation points does not increase the accuracy of integration.

Example 3. In work [11] as the second example solving an integral

$$I = \int_0^2 e^x \sin(50\cosh(x))dx,$$

initially introduced in work [14] was suggested.

In the same paper the exact value of the integral was given

$$I = 0.143079115028939 + i0.070765298796184.$$

We conducted integration on the Gauss-Lobatto grid with the number of nodes ranging from 39 to 49 using the methods described by us. We have solved the obtained System of linear equations by using LU decomposition method or SVD method [3] in all cases identical results with computer-accuracy were obtained (see Fig. 4). In the graph the numerical values of relative error compared with the exact value depending on the number of nodes on the grid are shown.

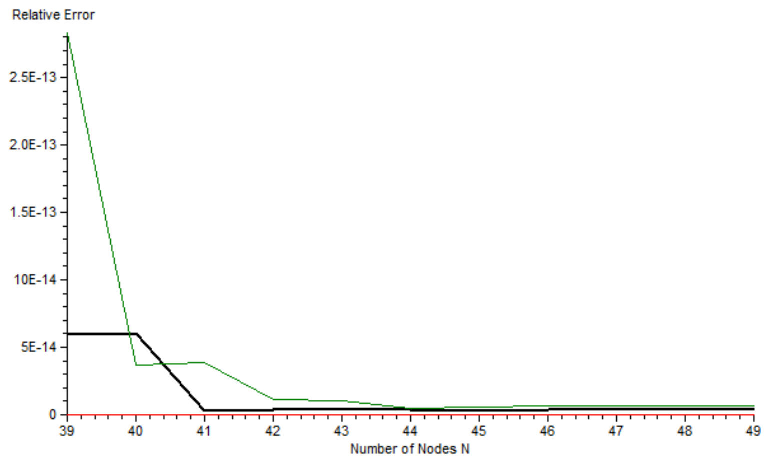

Figure 4: Relative error of computation depending of number of nodes: green line – imaginary part (minimum equals 4.96623859329411e-15, N=44), black (minimum equals 2.4626386140239e-15, N=41) – real part

The number of collocation points can be increased over adduced values, but such increase of the number of nodes does not lead to the calculation accuracy of integration.

7. Conclusion

Several practical ways of effective and sustainable solution of linear equation systems with the following solving of integrals from rapidly oscillating functions by using Levin's method for wide class of non-linear and even non-monotonic phase $g(x)$ functions were offered by us. Also an easy way of integration from rapidly oscillating functions in case of linear phase functions allowing to write out the value practically was offered, which allows getting the solution in almost analytical way.

Universal quadrature method for one-dimensional oscillating integrals allowing to obtain a very exact output computation of integral by reducing the task to the solution of linear algebraic equation sets is researched in this paper. This method leads to very accurate results even in case of degenerate linear algebraic equation sets. The advantage of this method is that the result can be reached with a small number (compared to direct numerical calculation technique) of sub-integral function calculations.

In the fifth section, we have proposed an alternative method of representation of the Chebyshev differentiation matrix based on the use of other recurrence relations. This differentiation matrix representation allows to demonstrate explicitly non-degeneracy of System of linear algebraic equations resulting from the Levin method in the cases when $g'(x) \neq 0$ in no grid points. Previously, it was shown that for the phase function with finite numbers of zeros of its derivative it is possible to choose such Gauss-Lobatto grid nodes and in a sustainable way to solve the resulting system of linear equations.

In the last section of the article the advantages of our approach are demonstrated by a number of numerical examples.

References

[1] Huybrechs D., Olver S. *Highly oscillatory quadrature*, in: B. Engquist, A. Fokas, E. Hairer, A. Iserles (eds.), Highly Oscillatory Problems, Cambridge Univ. Press, Cambridge, 2009, pp. 25–50.

[2] Jeffrey G. B., *Louis Napoleon George Filon. (1875-1937)*, Obituary Notices of Fellows of the Royal Society. 1939, **2**, no. 7, pp. 500–509.

[3] Golub G. H., Van Loan Ch. F. *Matrix computations. 4th ed.* 2013, Baltimore, JHU Press, xxi, 756 p. ill. ISBN 1421407949, 9781421407944.

[4] Levin D. *Procedures for computing one and two-dimensional integrals of functions with rapid irregular oscillations*, Math. Comp. 1982, **38**, no. 158, pp. 531–538.

[5] Iserles A. *On the numerical quadrature of highly-oscillatory integrals I: Fourier transforms*, IMA J. Num. Anal. 2004, **24**, pp. 1110–1123.

[6] Sevastianov L. A., Lovetsky K. P., Kokotchikova M.G. *Discrete Transformation of Mesh Functions Values to Fourier Polynomials Coefficients*. Bulletin of Peoples' Friendship University of Russia. Series “Mathematics. Information Scinces. Physics”. 2007, No 3–4, pp. 70–75.

[7] Sevastianov L. A., Kokotchikova M. G., Kulyabov D. S. *Data Processing by Method of Transformation of Functions and its Derivatives Values on Grids into Fourier Coefficients*. Bulletin of Peoples' Friendship University of Russia. Series “Mathematics. Information Scinces. Physics”. 2009, No 1, pp. 62–70.

[8] Mason J. C., Handscomb D. C. *Chebyshev Polynomials*, Chapman and Hall/CRC, 2002-09-17, 360 p.

[9] Lovetskiy K. P., Petrov V. V. *Integration of Highly Oscillatory Functions*. Bulletin of Peoples' Friendship University of Russia. Series "Mathematics. Information Scinces. Physics". 2011, No 2, pp. 92–97.

[10] Ahremenkov D. A., Lovetskiy K. P. *Solving Systems of Linear Differential Equations with Constant Coefficients*. Bulletin of Peoples' Friendship University of Russia. Series "Mathematics. Information Scinces. Physics". 2011, No 2, pp. 98–103.

[11] Jianbing L., XueSong W. and Tao W. *A universal solution to one-dimensional oscillatory integrals*. Sci China Ser. F-Inf. Sci. 2008, 51, pp. 1614–1622.

[12] Press, W. H., Teukolsky S. A., Vetterling W. T., Flannery B. P. Section 6.5. Bessel Functions of Integer Order, Numerical Recipes: The Art of Scientific Computing (3rd ed.). 2007, New York: Cambridge University Press.

[13] www.netlib.org/cephes , www.alglib.net

[14] Evans G. A., Webster J. R. *A comparison of some methods for the evaluation of highly oscillatory integrals*. J. Comp. Appl. Math. 1999, **112**, pp. 55–69.