http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 2, No 3, p. 28 – 47 (2014)

A perishable inventory model with two types of customers, Erlang-k service, linear repeated attempts and a finite populations

K. Jeganathan

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai - 600 005, India

e-mail: jegan.nathan@yahoo.com

Received 11 September 2014, in final form 19 October. Published 20 October 2014.

Abstract. The present paper deals with a generalization of the homogeneous single server finite source retrial queuing-inventory system with Erlang-k service in which the items may be serviced before it is delivered to the customers. We assume that not all customers would be requiring service on items. Hence we propose to have two types of customers, say, high priority and low priority. The high priority customer demands a unit item with require service on the demanded item before accepting it and the low priority customer demands a unit item but do not require any service on his demanded item (i.e., service time is zero). The service time of high priority customer follows an Erlang k-type distribution with service rate $k\mu$ for each phase. Retrial is introduced for low priority customers only. The life time of the item is assumed to have exponential distributions. The inventory is replenished according to an (s, S) policy and the replenishing times are assumed to be exponentially distributed. The joint probability distribution of the number of high priority customers in the waiting area, the number of low priority customers in the orbit and the inventory level is obtained for the steady state case. Some important system performance measures in the steady state are derived.

Keywords: (s, S) policy, Continuous review, Inventory with service time, Markov process, Linear repeated attempts, Priority customers, Finite population
MSC numbers: 60J27, 90B05

1. Introduction

Research on queueing systems with inventory control has captured much attention of researchers over the last decades. Many researchers assumed that customers arrive at the service facility one by one and require service. In order to complete the customer service, an item from the inventory is needed. A served customer departs immediately from the system and the on - hand inventory decreases by one at the moment of service completion. This type of system is called a queueing - inventory system [13]. Berman and Kim [2] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [5] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [6] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [3] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [4] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. The M/M/1 queueing - inventory system with backordering was investigated by Schwarz and Daduna [12]. Krishnamoorthy et al., [10] introduced an additional control policy (N-policy) into (s, S) inventory system with positive service time.

In a hairdresser's shop, shoe mart, show rooms, etc., a finite number of chairs are provided to waiting customers. All chairs might be occupied at arrival instant, and thus, an arriving customer is not able to enter the queue. In this case, the customer may not leave the system forever, but might retry to be served at another time. This type of system is called a retrial queueing - inventory system. Artalejo et al. [1] were the first to study inventory policies with positive lead time and retrial of customers who could not get service during their earlier attempts to access the service station; it may be noted that Ushakumari [15] obtained analytical solution to this model. Krishnamoorthy and Jose [9] analyzed and compared three (s, S) policies with positive service time and retrial of customers. They have assumed Poisson arrivals, exponential distributed lead times and retrial times. In that work, the authors proceeded with an algorithmic analysis of the system.

An important issue in the retrial queueing-inventory system with two classes of customers is the priority assignment problem. For example, in assembly manufacturing system customers with long-term supply contracts have been given high priority than the other ordinary customers. In multi-specialty hospitals patients with serious illness are given high priority than the other patients opting for routine check or else. The real life problems stimulate as to study the queueing-inventory systems with two types of customers. Ning Zhao and Zhaotong Lian [11] analyzed a queueing-inventory system with two classes of customers. The authors have assumed the arrival of the two-types of customers form independent Poisson processes and exponential service times. Each service uses one item from the attached inventory supplied by an outside supplier with exponentially distributed lead time. Choi and Chang [7] analyzed single server retrial queues with priority calls. Recently Jeganathan et al. [8] analyzed a retrial inventory system with non-preemptive priority service. The authors have assumed the arrival of the two-types of customers form independent Poisson processes and exponential service times. Retrial is introduced for low priority customers only.

In all these models, the authors assumed that an arriving customer should wait for the delivery of demanded items, as there is a random service time. But in some real life situations, the customers may accept their demanded items without any service performed on it. For example, in the case of personal computers, a unit is usually sold to a customer only after the assembly and installation. But a technically qualified engineer, may accept the kit of spares direct from the store as he can assemble it at a later time.

In this article, we investigate a finite-source retrial queueing inventory system with Erlang k service in which two types of customers, say, high priority and low priority customers. High priority customers require service on their demanding unit and low priority customers do not want service or negligible service on their demanding unit. Retrial is introduced for low priority customers only. The joint probability distribution of the number of high priority customers in the waiting area, the number of low priority customers in the orbit and the inventory level is obtained for the steady state case. Various measures of system performance are computed in the steady state case.

The rest of the paper is organized as follows. In the next section, the problem formulation and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are proposed in section 3. Some key system performance measures are derived in section 4. In section 5, we derived the total expected cost rate. The last section is meant for conclusion.

2. Problem formulation

Consider an inventory system with a maximum stock of S units and the demands originated from a finite population of sources $M = N_1 + N_2$. Customers arriving at the service station belong to any one of the two types such that the high priority and the low priority customers and their arrivals belong to independent quasi-random distributions with parameters λ_1 and λ_2 respectively. The high priority customers require service on their demanded unit and the low priority customers do not want any service to be performed on their demanded unit. The demand is for single item per customer. The waiting area is limited to accommodate a maximum number N_1 of high priority customers including the one at service point. Whenever the server is busy and only one item in the inventory or the inventory level is zero, an arriving primary low priority customer enters into an orbit of finite size N_2. Each high priority customer is either free or in the service facility at any time and each low priority customer is either free or in the orbit at any time.

There are two types of retrial policy considered in the literature of queuing systems:

1. The probability of a repeated attempt depends on the number of orbiting demands (classical retrial policy).

2. The probability of a repeated attempt is independent of the number of orbiting demands (constant retrial policy).

In this article we consider both types of retrial policy (classical retrial policy and constant retrial policy). More explicitly, when there are $j \geq 1$ low priority customers in the orbit, a signal is sent out according to an exponential distribution with parameter $\theta_j = \alpha(1 - \delta_{j0}) + j\nu$ when the orbit size is j and δ denotes the Kronecker delta. Note that the latter type of retrial policy is described as linear retrial policy. In the special case $\alpha = 0$ and $\nu > 0$ our model becomes the classical single-server retrial queuing-inventory system. Alternatively, when $\nu = 0$ and $\alpha > 0$ our model becomes the constant retrial queueing-inventory system.

Let k be the number of phases in the service station. Assume that the service time has Erlang-k type distribution with service rate $k\mu$ for each phase. After the lost (kth phase of service) service completed on the item, the item is delivered to the customer by the server. The life time of each item has negative exponential distribution with parameter $\gamma > 0$. We have assumed that an item of inventory that makes it into the service process cannot perish while in service. Any arriving high priority customer, who finds the waiting area full is considered to be lost. It is also assumed that any arriving primary low priority customer, who finds that either the server is busy with only one item in the inventory and orbit size is full or inventory level is zero with no space in the orbit, is considered to be lost. The reorder level for the commodity is fixed as s and an order is placed when the inventory level reaches the reorder level s. The ordering quantity for the commodity is $Q(= S - s > s + 1)$ items. The requirement $S - s > s + 1$ ensures that after a replenishment the inventory level will be always above the reorder level. Otherwise it may not be possible to place reorder which leads to perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter $\beta(> 0)$. Further, it is assumed that the inter arrival times between high and low priority customers, intervals between repeated attempts of the retrial times, service time of high priority customers, the lead times and the life time of each items are mutually independent random variables.

2.1 Notations

$$
\begin{array}{rcl}
\mathbf{e} & : & \text{a column vector of appropriate dimension containing all ones} \\
\mathbf{0} & : & \text{Zero matrix} \\
[A]_{ij} & : & \text{entry at } (i,j)^{th} \text{ position of a matrix A} \\
\delta_{ij} & : & \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{otherwise} \end{cases} \\
\bar{\delta}_{ij} & : & 1 - \delta_{ij} \\
H(x) & : & \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \\
k \in V_i^j & : & k = i, i+1, \ldots j \\
M & : & N_1 + N_2
\end{array}
$$

2.2 Classification of states

The states of the system are divided into classes as follows:

1. $E_a = \{E_{a0} = (0, 0, i_3, i_4) \mid i_3 = 0, 1, 2, \ldots, N_1,\ i_4 = 0, 1, 2, \ldots, N_2, \}$ where E_{a0} is the state in which no items in the inventory ($i_1 = 0$), service has not begin (i.e., server is idle), number of high priority customers in the waiting area lies between zero to N_1 and number of low priority customers in the orbit lies between zero to N_2.

2. $E_b = \{E_{b0} = (i_1, 0, 0, i_4) \mid i_1 = 1, 2, \ldots, S,\ i_4 = 0, 1, 2, \ldots, N_2, \}$ where E_{b0} is the state in which i_1 items are in the inventory, server is idle (i.e., $i_2 = 0$), no high priority customers in the waiting area and i_4 low priority customers are in the orbit.

3. $E_c = \{E_{c0} = (i_1, i_2, i_3, i_4) \mid i_1 = 1, 2, \ldots, S,\ i_2 = 1, 2, \ldots, k,\ i_3 = 1, 2, \ldots, N_1,$ $i_4 = 0, 1, 2, \ldots, N_2, \}$ where E_{c0} is the state in which i_1 items are in the inventory, i_2 means that the server is busy with the customer in the ${i_2}^{th}$ phase, i_3 means the number of high priority customers in the waiting area and i_4 means the number of low priority customers are in the orbit.

3. Analysis

Let $L(t)$, $Y(t)$, $X(t)$ and $Z(t)$ respectively, denote the inventory level, the server status, the number of high priority customers in the waiting area and the number of low priority customers in the orbit at time t.

Further, server status $Y(t)$ be defined as follows:

$$Y(t):\begin{cases} 0, & \text{if the server is idle at time t,} \\ 1, & \text{if the server is providing first phase service to a high priority customer} \\ & \text{at time t,} \\ 2, & \text{if the server is providing second phase service to a high priority customer} \\ & \text{at time t,} \\ \vdots & \vdots \\ k, & \text{if the server is providing } k^{th} \text{ phase service to a high priority customer} \\ & \text{at time t,} \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t) = \{(L(t), Y(t), X(t), Z(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by $E = E_a \cup E_b \cup E_c$.

To determine the infinitesimal generator

$$\Theta = ((\ d((i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4))\ )), \quad (i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4) \in E$$

of this process we use the following arguments :

- We first note that in a Markov process there can be at most one change in the levels of the state of the process through any one of the activities - arrival (high priority/low priority), failure of an item, replenishment of stock and change of phase of service.
- If the server is idle and the inventory level is zero,
  - any arriving high priority customer takes the state of the process from $(0, 0, i_3, i_4)$ to $(0, 0, i_3{+}1, i_4)$ with the intensity of transition $d((0, 0, i_3, i_4), (0, 0, i_3 + 1, i_4))$ is given by $(M - i_3 - i_4)\lambda_1$, $i_3 = 0, 1, 2, \ldots, N_1 - 1$, $i_4 = 0, 1, 2, \ldots, N_2$.
  - any arriving primary low priority customer takes the state of the process from $(0, 0, i_3, i_4)$ to $(0, 0, i_3, i_4 + 1)$ with the intensity of transition $d((0, 0, i_3, i_4), (0, 0, i_3, i_4 + 1))$ is given by $(M - i_3 - i_4)\lambda_2$, $i_3 = 0, 1, 2, \ldots, N_1$, $i_4 = 0, 1, 2, \ldots, N_2 - 1$.
- If the server is idle and the inventory level lies between one to S,
  - an arrival of high priority customer joins the waiting area and immediately taken for first phase service by the server. The state of the process from $(i_1, 0, 0, i_4)$ to $(i_1, 1, 1, i_4)$ and the intensity of this transition $d((i_1, 0, 0, i_4), (i_1, 1, 1, i_4))$ is given by $(M - i_4)\lambda_1$, $i_1 = 1, 2, \ldots, S$, $i_4 = 0, 1, 2, \ldots, N_2$.

- a transition from state $(i_1, 0, 0, i_4)$ to state $(i_1 - 1, 0, 0, i_4)$ takes place when any one of "i_1" items perishes for which the rate is $i_1\gamma$ or when a primary low priority customer from any one of the $(M - i_4)$ sources occurs for which the rate is $(M - i_4)\lambda_2$. Hence, the intensity of this transition is $i_1\gamma + (M - i_4)\lambda_2$, where $i_1 = 1, 2, \ldots, S$, $i_4 = 0, 1, 2, \ldots, N_2$.
- any arriving retrial customer takes the state of the process from $(i_1, 0, 0, i_4)$ to $(i_1-1, 0, 0, i_4-1)$ and the intensity of this transition $d((i_1, 0, 0, i_4), (i_1-1, 0, 0, i_4 - 1))$ is given by $\theta_{i_4} = \alpha(1 - \delta_{i_4 0}) + i_4\nu$, $i_1 = 1, 2, \ldots, S$, $i_4 = 1, 2, \ldots, , N_2$.

- If the server is busy and the inventory level lies between one to S,
  - a transition from (i_1, i_2, i_3, i_4) to $(i_1, i_2, i_3 + 1, i_4)$ $i_1 = 1, 2, \ldots, S$, $i_2 = 1, 2, \ldots, k$, $i_3 = 1, 2, \ldots, N_1 - 1$, $i_4 = 0, 1, 2, \ldots, N_2$, will take place with intensity of transition $(M - i_3 - i_4)\lambda_1$, when a high priority customer arrives.
  - the arrival of a primary low priority customer enters into the orbit. Hence a transition takes place from (i_1, i_2, i_3, i_4) to $(i_1, i_2, i_3, i_4 + 1)$ with intensity $(M - i_3 - i_4)\lambda_2$, $i_1 = 1$, $i_2 = 1, 2, \ldots, k$, $i_3 = 1, 2, \ldots, , N_1$, $i_4 = 0, 1, 2, \ldots, N_2 - 1$.
  - a transition from state (i_1, i_2, i_3, i_4) to state $(i_1 - 1, i_2, i_3, i_4)$ takes place when any one of $(i_1 - 1)$ items perishes for which the rate is $(i_1 - 1)\gamma$ or when a primary low priority customer from any one of the $(M - i_3 - i_4)$ sources occurs for which the rate is $(M - i_3 - i_4)\lambda_2$. Hence, the intensity of this transition is $(i_1 - 1)\gamma + (M - i_3 - i_4)\lambda_2$, where $i_1 = 2, \ldots, S$, $i_2 = 1, 2, \ldots, k$, $i_3 = 1, 2, \ldots, N_1$, $i_4 = 0, 1, 2, \ldots, N_2 - 1$.
  - a retrial customer takes the state of the process from (i_1, i_2, i_3, i_4) to (i_1-1, i_2, i_3, i_4-1) and the intensity of this transition is $\theta_{i_4} = \alpha(1-\delta_{i_4 0})+i_4\nu$, $i_1 = 2, \ldots, S$, $i_2 = 1, 2, \ldots, k$, $i_3 = 1, 2, \ldots, N_1$, $i_4 = 1, 2, \ldots, , N_2$.
  - the completion of any one the i_2th phase service for a high priority customer makes a transition from (i_1, i_2, i_3, i_4) to $(i_1, i_2 + 1, i_3, i_4)$ with intensity of transition $k\mu$, $i_1 = 1, 2, \ldots, S$, $i_2 = 1, 2, \ldots, k - 1$, $i_3 = 1, 2, \ldots, N_1$, $i_4 = 1, 2, \ldots, , N_2$.
  - a transition from (i_1, k, i_3, i_4) to $(i_1 - 1, 1, i_3 - 1, i_4)$, $i_1 = 1, 2, \ldots, S$, $i_3 = 1, 2, \ldots, N_1$, $i_4 = 0, 1, 2, \ldots, , N_2$ with intensity of transition $k\mu$, when the server completes kth phase service for a high priority customer.

- A passage from $(i_1, 0, 0, i_4)$ to $(i_1 + Q, 0, 0, i_4)$, where $Q(= S - s)$, for $i_1 = 0, 1, 2, \ldots, s$, $i_4 = 0, 1, 2, \ldots, N_2$, or from $(0, 0, i_3, i_4)$ to $(Q, 1, i_3, i_4)$ for $i_3 = 1, 2, \ldots, N_1$, $i_4 = 0, 1, 2, \ldots, , N_2$ or from (i_1, i_2, i_3, i_4) to $(i_1 + Q, i_2, i_3, i_4)$ for $i_1 = 0, 1, 2, \ldots, s$, $i_2 = 1, 2, \ldots, k$, $i_3 = 1, 2, \ldots, N_1$, $i_4 = 0, 1, 2, \ldots, , N_2$, will take place with intensity of transition β when a replenishment for Q items occurs.

- For other transition from (i_1, i_2, i_3, i_4) to (j_1, j_2, j_3, j_4), except $(i_1, i_2, i_3, i_4) \neq (j_1, j_2, j_3, j_4)$, the rate is zero.
- Finally, note that

$$d((i_1, i_2, i_3, i_4), (i_1, i_2, i_3, i_4)) = - \sum_{j_1} \sum_{j_2} \sum_{j_3} \sum_{\substack{j_4 \\ (j_1,j_2,j_3,j_4) \neq (i_1,i_2,i_3,i_4)}} d((i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4)).$$

Hence, we have $d((i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4)) =$

$$= \begin{cases}
\beta, & \begin{array}{llll} j_1 = Q, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4, \\ i_1 = 0, & i_2 = 0, & i_3 = 0, & i_4 \in V_0^{N_2}, \\ & \text{or} & & \\ j_1 = Q, & j_2 = 1, & j_3 = i_3, & j_4 = i_4, \\ i_1 = 0, & i_2 = 0, & i_3 \in V_1^{N_1}, & i_4 \in V_0^{N_2}, \\ & \text{or} & & \\ j_1 = i_1 + Q, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4, \\ i_1 \in V_1^{s}, & i_2 \in V_0^{k}, & i_3 \in V_0^{N_1}, & i_4 \in V_0^{N_2}, \end{array} \\[1em]
i_1\gamma + (M - i_4)\lambda_2, & \begin{array}{llll} j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4, \\ i_1 \in V_1^{S}, & i_2 = 0, & i_3 = 0, & i_4 \in V_0^{N_2}, \end{array} \\[1em]
\begin{array}{l} (i_1 - 1)\gamma + \\ (M - i_3 - i_4)\lambda_2, \end{array} & \begin{array}{llll} j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4, \\ i_1 \in V_2^{S}, & i_2 \in V_1^{k}, & i_3 \in V_1^{N_1}, & i_4 \in V_0^{N_2}, \end{array} \\[1em]
\theta_{i_4}, & \begin{array}{llll} j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4 - 1, \\ i_1 \in V_1^{S}, & i_2 = 0, & i_3 = 0, & i_4 \in V_1^{N_2}, \\ & \text{or} & & \\ j_1 = i_1 - 1, & j_2 = i_2, & j_3 = i_3, & j_4 = i_4 - 1, \\ i_1 \in V_2^{S}, & i_2 \in V_1^{k}, & i_3 \in V_1^{N_1}, & i_4 \in V_1^{N_2}, \end{array} \\[1em]
k\mu, & \begin{array}{llll} j_1 = i_1 - 1, & j_2 = 0, & j_3 = i_3 - 1, & j_4 = i_4, \\ i_1 = 1, & i_2 = k, & i_3 \in V_1^{N_1}, & i_4 \in V_0^{N_2}, \\ & \text{or} & & \\ j_1 = i_1 - 1, & j_2 = 0, & j_3 = i_3 - 1, & j_4 = i_4, \\ i_1 \in V_2^{S}, & i_2 = k, & i_3 = 1, & i_4 \in V_0^{N_2}, \\ & \text{or} & & \\ j_1 = i_1 - 1, & j_2 = 1, & j_3 = i_3 - 1, & j_4 = i_4, \\ i_1 \in V_2^{S}, & i_2 = k, & i_3 \in V_2^{N_1}, & i_4 \in V_0^{N_2}, \\ & \text{or} & & \\ j_1 = i_1, & j_2 = i_2 + 1, & j_3 = i_3, & j_4 = i_4, \\ i_1 \in V_1^{S}, & i_2 \in V_1^{k-1}, & i_3 \in V_1^{N_1}, & i_4 \in V_0^{N_2}, \end{array}
\end{cases}$$

$$
\begin{cases}
(M-i_3-i_4)\lambda_1, & \begin{array}{llll} j_1=i_1, & j_2=i_2, & j_3=i_3+1, & j_4=i_4, \\ i_1=0, & i_2=0, & i_3\in V_0^{N_1-1}, & i_4\in V_0^{N_2-1}, \\ & \text{or} & & \\ j_1=i_1, & j_2=1, & j_3=1, & j_4=i_4, \\ i_1\in V_1^S, & i_2=0, & i_3=0, & i_4\in V_0^{N_2}, \\ & \text{or} & & \\ j_1=i_1, & j_2=i_2, & j_3=i_3+1, & j_4=i_4, \\ i_1\in V_1^S, & i_2\in V_1^k, & i_3\in V_1^{N_1-1}, & i_4\in V_0^{N_2}, \end{array} \\
(M-i_3-i_4)\lambda_2, & \begin{array}{llll} j_1=i_1, & j_2=i_2, & j_3=i_3, & j_4=i_4+1, \\ i_1=0, & i_2=0, & i_3\in V_0^{N_1}, & i_4\in V_0^{N_2}, \\ & \text{or} & & \\ j_1=i_1, & j_2=i_2, & j_3=i_3, & j_4=i_4+1, \\ i_1=1, & i_2\in V_1^k, & i_3\in V_1^{N_1}, & i_4\in V_0^{N_2-1}, \end{array} \\
\begin{array}{l} -((M-i_3-i_4)\times \\ (\lambda_1+\bar{\delta}_{i_4N_2}\lambda_2)+\beta), \end{array} & \begin{array}{llll} j_1=i_1, & j_2=i_2, & j_3=i_3, & j_4=i_4, \\ i_1=0, & i_2=0, & i_3\in V_0^{N_1}, & i_4\in V_0^{N_2}, \end{array} \\
\begin{array}{l} -((M-i_3-i_4)(\lambda_1+\lambda_2)+ \\ H(s-i_1)\beta+i_1\gamma+\theta_{i_4}+k\mu), \end{array} & \begin{array}{llll} j_1=i_1, & j_2=i_2, & j_3=i_3, & j_4=i_4, \\ i_1\in V_1^S, & i_2=0, & i_3=0, & i_4\in V_0^{N_2}, \end{array} \\
\begin{array}{l} -((M-i_3-i_4)(\bar{\delta}_{i_3N_1}\lambda_1+ \\ \lambda_2)+k\mu+H(s-i_1)\beta+ \\ (i_1-1)\gamma+\bar{\delta}_{i_11}\theta_{i_4}), \end{array} & \begin{array}{llll} j_1=i_1, & j_2=i_2, & j_3=i_3, & j_4=i_4, \\ i_1\in V_1^S, & i_2\in V_1^k, & i_3\in V_1^{N_1}, & i_4\in V_0^{N_2} \end{array} \\
0, & \text{Otherwise.}
\end{cases}
$$

Define the following ordered sets:

$$
\begin{aligned}
< i_1,i_2,i_3 > &= \begin{cases} (i_1,0,i_3,i_4), & i_1=0;\ i_3=0,1,2,\dots,N_1;\ i_3=0,1,2,\dots,N_2; \\ (i_1,0,0,i_4), & i_1=1,2,\dots,S;\ i_4=0,1,2,\dots,N_2; \\ (i_1,i_2,i_3,i_4), & i_1=1,2,\dots,S;\ i_2=1,2,\dots,k;\ i_3=1,2,\dots,N_1; \\ & i_4=0,1,2,\dots,N_2; \end{cases} \\
\ll i_1,i_2 \gg &= \begin{cases} < i_1,0,i_3 >, & i_1=0;\ i_3=0,1,2,\dots,N_1; \\ < i_1,0,i_3 >, & i_1=1,2,\dots,S;\ i_3=0; \\ < i_1,i_2,i_3 >, & i_1=1,2,\dots,S;\ i_2=1,2,\dots,k;\ i_3=1,2,\dots,N_1 \end{cases} \\
\lll i_1 \ggg &= \begin{cases} \ll i_1,0,\gg, & i_1=0,1,2,\dots S; \\ \ll i_1,i_2\gg, & i_1=1,2,\dots,S;\ i_2=1,2,\dots,k; \end{cases}
\end{aligned}
$$

By ordering the sets of state space as $(\lll 1 \ggg, \lll 2 \ggg, \dots, \lll S \ggg)$, the infinitesimal generator Θ can be conveniently expressed in a block partitioned matrix with entries

$$\Theta_{i_1 j_1} = \begin{cases} A_{i_1} & j_1 = i_1,\ i_1 = 0, 1, 2, \dots, S \\ B_{i_1} & j_1 = i_1 - 1,\ i_1 = 1, 2, \dots, S-1, S \\ C & j_1 = i_1 + Q,\ i_1 = 1, 2, \dots, s, \\ C_1 & j_1 = i_1 + Q,\ i_1 = 0, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

More explicitly,

$$\Theta = \begin{array}{c} S \\ S-1 \\ \vdots \\ s+1 \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array} \begin{pmatrix} A_S & B_S & & & & & & & & \\ & A_{S-1} & B_{S-1} & & & & & & & \\ & & \cdots & & & & & & & \\ & & & \cdots & A_{s+1} & B_{s+1} & & & & \\ C & & & & & A_s & B_s & & & \\ & C & & & & & A_{s-1} & & & \\ & & \cdots & & & & \cdots & & & \\ & & C & & & & & \cdots & A_1 & B_1 \\ & & & C_1 & & & & & & A_0 \end{pmatrix}$$

where

$$[C_1]_{i_2 j_2} = \begin{cases} C_{00}^{(1)} & j_2 = i_2, \quad i_2 = 0, \\ C_{01}^{(1)} & j_2 = 1, \quad i_2 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_{00}^{(1)}]_{i_3 j_3} = \begin{cases} W_0 & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_{01}^{(1)}]_{i_3 j_3} = \begin{cases} W_0 & j_3 = i_3, \quad i_3 \in V_1^{N_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[W_0]_{i_4 j_4} = \begin{cases} \beta, & j_4 = i_4, \quad i_4 \in V_0^{N_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[C]_{i_2 j_2} = \begin{cases} C_{00}^{(0)} & j_2 = i_2, \quad i_2 = 0, \\ C_{11} & j_2 = 1, \quad i_2 \in V_1^k \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_{00}^{(0)}]_{i_3 j_3} = \begin{cases} W_0 & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_{11}]_{i_3 j_3} = \begin{cases} W_0 & j_3 = i_3, \quad i_3 \in V_1^{N_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_1]_{i_2j_2} = \begin{cases} B_{00}^{(1)} & j_2 = i_2, \quad i_2 = 0, \\ B_{k0}^{(1)} & j_2 = 0, \quad i_2 = k, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_{00}{}^{(1)}]_{i_3j_3} = \begin{cases} K_1 & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[K_1]_{i_4j_4} = \begin{cases} (\gamma + (M - i_4)\lambda_2), & j_4 = i_4, \quad i_4 \in V_0^{N_2} \\ \theta_{i_4}, & j_4 = i_4 - 1, \quad i_4 \in V_1^{N_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[B_{k0}{}^{(1)}]_{i_3j_3} = \begin{cases} F_0 & j_3 = i_3 - 1, \quad i_3 \in V_1^{N_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[F_0]_{i_4j_4} = \begin{cases} k\mu, & j_4 = i_4, \quad i_4 \in V_0^{N_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[B_{k0}]_{i_3j_3} = \begin{cases} F_0 & j_3 = 0, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_{K1}]_{i_3j_3} = \begin{cases} F_0 & j_3 = i_3 - 1, \quad i_3 \in V_2^{N_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i_1 = 2, 3, \ldots, S$

$$[B_{i_1}]_{i_2j_2} = \begin{cases} B_{00}^{(i_1)} & j_2 = 0, \quad i_2 = 0 \\ U_{i_1} & j_2 = i_2, \quad i_2 \in V_1^k \\ B_{k0}^{(i_1)} & j_2 = 0, \quad i_2 = k \\ B_{k1}^{(i_1)} & j_2 = 1, \quad i_2 = k \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_{00}^{(i_1)}]_{i_3j_3} = \begin{cases} K_{i_1} & j_3 = i_3, \quad i_3 = 0 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[K_{i_1}]_{i_4j_4} = \begin{cases} (i_1\gamma + (M - i_4)\lambda_2), & j_4 = i_4, \quad i_4 \in V_0^{N_2} \\ \theta_{i_4}, & j_4 = i_4 - 1, \quad i_4 \in V_1^{N_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[U_{i_1}]_{i_3j_3} = \begin{cases} V_{i_1-1,i_3} & j_3 = i_3, \quad i_3 \in V_1^{N_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i_1 = 2, 3, \ldots, S,\ i_3 = 1, 2, 3, \ldots, N_1$

$$[V_{i_1-1,i_3}]_{i_4j_4} = \begin{cases} (i_1-1)\gamma+(M-i_3-i_4)\lambda_2, & j_4=i_4, \quad i_4\in V_0^{N_2} \\ \theta_{i_4}, & j_4=i_4-1, \quad i_4\in V_1^{N_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[A_0]_{i_2j_2} = \begin{cases} A_{00}^{(0)} & j_2=0, \quad i_2=0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[A_{00}^{(0)}]_{i_3j_3} = \begin{cases} G_{i_3} & j_3=i_3+1, \quad i_3\in V_0^{N_1-1} \\ H_{i_3} & j_3=i_3, \quad i_3\in V_0^{N_1}, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i_3=0,1,2,\ldots,N_1-1$

$$[G_{i_3}]_{i_4j_4} = \begin{cases} (M-i_3-i_4)\lambda_1 & j_4=i_4, \quad i_4\in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

For $i_3=0,1,2,\ldots,N_1$

$$[H_{i_3}]_{i_4j_4} = \begin{cases} (M-i_3-i_4)\lambda_2 & j_4=i_4+1, \quad i_4\in V_0^{N_2-1}, \\ -((M-i_3-i_4)(\lambda_1+\bar{\delta}_{i_4N_2}+\beta) & j_4=i_4, \quad i_4\in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$[A_1]_{i_2j_2} = \begin{cases} A_{00}^{(1)} & j_2=0, \quad i_2=0, \\ A_{00} & j_2=1, \quad i_2=0, \\ L_0 & j_2=i_2, \quad i_2\in V_1^{k}, \\ E_0 & j_2=i_2+1, \quad i_2\in V_1^{k-1}, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[A_{00}]_{i_3j_3} = \begin{cases} G_0 & j_3=1, \quad i_3=0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[J_0]_{i_4j_4} = \begin{cases} -((M-i_3-i_4)(\lambda_1+\lambda_2)+ & \\ \gamma+\theta_{i_4}+\beta), & j_4=i_4, \quad i_4\in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$[L_0]_{i_3j_3} = \begin{cases} G_{i_3} & j_3=i_3+1, \quad i_3\in V_1^{N_1-1}, \\ D_{1,i_3} & j_3=i_3, \quad i_3\in V_1^{N_1-1}, \\ D & j_3=i_3, \quad i_3=N_1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[D_{1,i_3}]_{i_4j_4} = \begin{cases} -((M-i_3-i_4)(\bar{\delta}_{i_3N_1}\lambda_1+\bar{\delta}_{i_4N_2}\lambda_2)+ & \\ k\mu) & j_4=i_4, \quad i_4\in V_0^{N_2}, \\ (M-i_3-i_4)(\bar{\delta}_{i_4N_2}\lambda_2) & j_4=i_4+1, \quad i_4\in V_0^{N_2-1}, \\ 0, & \text{otherwise.} \end{cases}$$

$$[D]_{i_4i_4} = \begin{cases} (M-i_3-i_4)(\bar{\delta}_{i_4N_2}\lambda_2) & j_4=i_4+1, \quad i_4\in V_0^{N_2-1}, \\ -((M-i_3-i_4)\bar{\delta}_{i_4N_2}\lambda_2+ & \\ k\mu+\beta), & j_4=i_4, \quad i_4\in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

For $i_1 = 2, 3, \ldots, S$

$$[A_{i_1}]_{i_2 j_2} = \begin{cases} A_{00} & j_2 = 0, \quad i_2 = 0, \\ L_{i_1} & j_2 = i_2, \quad i_2 = 0, \\ L_{(i_1-1)} & j_2 = i_2, \quad i_2 \in V_1^k, \\ E_0 & j_2 = i_2 + 1, \quad i_2 \in V_1^{k-1}, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[L_{i_1}]_{i_3 j_3} = \begin{cases} J_{i_1} & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[J_{i_1}]_{i_4 j_4} = \begin{cases} -((M - i_3 - i_4)(\lambda_1 + \lambda_2) + i_1\gamma + \\ +\theta_{i_4} + H(s - i_1)\beta) & j_4 = i_4, \quad i_4 \in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

$$[L_{(i_1-1)}]_{i_3 j_3} = \begin{cases} G_{i_3} & j_3 = i_3 + 1, \quad i_3 \in V_0^{N_1-1}, \\ R_{(i_1-1, i_3)} & j_3 = i_3, \quad i_3 \in V_1^{N_1}, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

For $i_3 = 1, 2, 3, \ldots, N_1$,

$$[R_{(i_1-1, i_3)}]_{i_4 i_4} = \begin{cases} -(M - i_3 - i_4)(\bar{\delta}_{i_3 N_1}\lambda_1 + \lambda_2) + \\ \theta_{i_4} + k\mu + H(s - i_1)\beta), & j_4 = i_4, \quad i_4 \in V_0^{N_2}, \\ 0, & \text{otherwise.} \end{cases}$$

It can be noted that A_{i_1}, B_{i_1}, $i_1 = 2, \ldots, S$, A_1, and C are square matrices of order $(N_2+1)(kN_1+1)$. C_1 is of size $(N_1+1)(N_2+1) \times (N_2+1)(kN_1+1)$, A_0 and A_{00}^0 are square matrices of order $(N_1+1)(N_2+1)$, B_1 is of size $(N_2+1)(kN_1+1) \times (N_1+1)(N_2+1)$. The sub matrices C_{00}^1, W_0, C_{00}^0, K_1, K_{i_1}, $B_{00}^{(i_1)}, i_1 = 1, 2, \ldots, S$, $V_{i_1-1,i_3}, i_3 = 1, 2, \ldots, N_1$, $G_{i_1}, i_3 = 0, 1, 2, \ldots, N_1 - 1$, $H_{i_3}, i_3 = 0, 1, 2, \ldots, N_1$, A_{00}^1, J_0, $D_{1,i_3}, i_3 = 0, 1, 2, \ldots, N_1 - 1$, D, L_{i_1}, S, J_{i_1}, $R_{(i_1-1,i_3)}, i_1 = 2, 3, \ldots, S, i_3 = 0, 1, 2, \ldots, N_1$ are square matrices of order (N_2+1). C_{11}, U_{i_1}, $B_{k1}^{(i_1)}$, $L_{(i_1-1)}, i_1 = 2, 3, \ldots, S$, E_0, L_0, are square matrices of order $N_1(N_2+1)$. $C_{01}^{(1)}$, $B_{k0}^{(1)}$, $B_{k0}^{(i_1)}, i_1 = 2, 3, \ldots, S$, and A_{00} are matrices of order $(N_1+1)(N_2+1) \times N_1(N_2+1)$, $N_1(N_2+1) \times (N_1+1)(N_2+1)$, $N_1(N_2+1) \times (N_2+1)$ and $(N_2+1) \times N_1(N_2+1)$ respectively.

3.1 Steady State Analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X(t), Z(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3,i_4)} = \lim_{t\to\infty} Pr[L(t)=i_1, Y(t)=i_2, X(t)=i_3, Z(t)=i_4 | L(0), Y(0), X(0), Z(0)]$$

exists.

Let $\mathbf{\Phi}$ denote the steady state probability vector of the generator Θ. The vector, $\mathbf{\Phi}$, partitioned as $\mathbf{\Phi} = (\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \dots, \mathbf{\Phi^{(S)}})$, where

$$\mathbf{\Phi^{(0)}} = (\mathbf{\Phi^{(0,0)}}), \qquad \mathbf{\Phi^{(i_1)}} = (\mathbf{\Phi^{(i_1,0)}}, \mathbf{\Phi^{(i_1,1)}}, \dots, \mathbf{\Phi^{(i_1,k)}}), \qquad i_1 = 1,2,3,\dots,S;$$

which is partitioned as follows:

$$\begin{aligned}
\mathbf{\Phi^{(0,0)}} &= (\phi^{\mathbf{(0,0,0)}}, \phi^{\mathbf{(0,0,1)}}, \dots, \phi^{\mathbf{(0,0,N_1)}}), \\
\mathbf{\Phi^{(i_1,0)}} &= (\phi^{\mathbf{(i_1,0,0)}}), \qquad\qquad\qquad\qquad i_1 = 1,2,3,\dots,S; \\
\mathbf{\Phi^{(i_1,i_2)}} &= (\phi^{\mathbf{(i_1,i_2,1)}}, \phi^{\mathbf{(i_1,i_2,2)}}, \dots, \phi^{\mathbf{(i_1,i_2,N_1)}}), \; i_1 = 1,2,3,\dots,S; \; i_2 = 1,2,\dots k,
\end{aligned}$$

Further the above vectors also partitioned as follows:

$$\begin{aligned}
\mathbf{\Phi^{(0,0,i_3)}} &= (\phi^{(0,0,i_3,0)}, \phi^{(0,0,i_3,1)}, \dots, \phi^{(0,0,i_3,N_2)}), \; i_3 = 0,1,2,3,\dots,N_1; \\
\mathbf{\Phi^{(i_1,0,0)}} &= (\phi^{(i_1,0,0,0)}, \phi^{(i_1,0,0,1)}, \dots, \phi^{(i_1,0,0,N_2)}), \; i_1 = 1,2,3,\dots,S; \\
\mathbf{\Phi^{(i_1,i_2,i_3)}} &= (\phi^{(i_1,i_2,i_3,0)}, \phi^{(i_1,i_2,i_3,1)}, \dots, \phi^{(i_1,i_2,i_3,N_2)}), \; i_1 = 1,2,\dots,S; \\
& \quad i_2 = 1,2,\dots,k; i_3 = 0,1,\dots,N_1;
\end{aligned}$$

The computation of steady state probability vector $\mathbf{\Phi} = (\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \dots, \mathbf{\Phi^{(S)}})$, by solving the following set of equations

$$\begin{aligned}
\mathbf{\Phi}^{\boldsymbol{i_1}} B_{i_1} + \mathbf{\Phi}^{\boldsymbol{i_1-1}} A_{i_1-1} &= \mathbf{0}, \qquad i_1 = 1,2,\dots,Q, \\
\mathbf{\Phi}^{\boldsymbol{i_1}} B_{i_1} + \mathbf{\Phi}^{\boldsymbol{i_1-1}} A_{i_1-1} + \mathbf{\Phi}^{\boldsymbol{(i_1-1-Q)}} C_1 &= \mathbf{0}, \qquad i_1 = Q+1, \\
\mathbf{\Phi}^{\boldsymbol{i_1}} B_{i_1} + \mathbf{\Phi}^{\boldsymbol{i_1-1}} A_{i_1-1} + \mathbf{\Phi}^{\boldsymbol{(i_1-1-Q)}} C &= \mathbf{0}, \qquad i_1 = Q+2, Q+3, \dots, S, \\
\mathbf{\Phi}^{\boldsymbol{S}} A_S + \mathbf{\Phi}^{\boldsymbol{s}} C &= \mathbf{0},
\end{aligned}$$

subject to conditions $\mathbf{\Phi}\Theta = \mathbf{0}$ and $\sum\sum\sum_{(i_1,i_2,i_3)} \phi^{(i_1,i_2,i_3)} = 1$.

This is done by the following algorithm.

Step 1. Solve the following system of equations to find the value of $\mathbf{\Phi}^{\boldsymbol{Q}}$

$$\begin{aligned}
\mathbf{\Phi}^{\boldsymbol{Q}} & \left[\left\{ (-1)^Q \sum_{j=0}^{s-1} \left[\left(\underset{k=Q}{\overset{s+1-j}{\Omega}} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\underset{l=S-j}{\overset{Q+2}{\Omega}} B_l A_{l-1}^{-1} \right) \right] \right\} B_{Q+1} \right. \\
& \left. + A_Q + \left\{ (-1)^Q \underset{j=Q}{\overset{1}{\Omega}} B_j A_{j-1}^{-1} \right\} C \right] = \mathbf{0},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{\Phi}^{\boldsymbol{Q}} & \left[\sum_{i_1=0}^{Q-1} \left((-1)^{Q-i_1} \underset{j=Q}{\overset{i_1+1}{\Omega}} B_j A_{j-1}^{-1} \right) + I \right. \\
& \left. + \sum_{i_1=Q+1}^{S} \left((-1)^{2Q-i_1+1} \sum_{j=0}^{S-i_1} \left[\left(\underset{k=Q}{\overset{s+1-j}{\Omega}} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\underset{l=S-j}{\overset{i_1+1}{\Omega}} B_l A_{l-1}^{-1} \right) \right] \right) \right] \boldsymbol{e} \; = \; 1.
\end{aligned}$$

Step 2. Compute the values of

$$
\begin{aligned}
\mathbf{\Omega}_{i_1} &= (-1)^{Q-i_1}\mathbf{\Phi}^{\mathbf{Q}} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1}, && i_1 = Q-1, Q-2, \dots, 0 \\
&= (-1)^{2Q-i_1+1}\mathbf{\Phi}^{\mathbf{Q}} \sum_{j=0}^{S-i_1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right], \\
& && i_1 = S, S-1, \dots, Q+1 \\
&= I, && i_1 = Q
\end{aligned}
$$

Step 3. Using $\mathbf{\Phi^{(Q)}}$ and $\Omega_{i_1}, i_1 = 0, 1, \dots, S$, calculate the value of $\mathbf{\Phi^{(i_1)}}, i_1 = 0, 1, \dots, S$. That is,

$$
\mathbf{\Phi^{(i_1)}} = \mathbf{\Phi^{(Q)}}\Omega_{i_1}, \quad i_1 = 0, 1, \dots, S.
$$

4. System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

4.1 Expected Inventory Level

Let η_I denote the excepted inventory level in the steady state. Since $\Phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of high priority customers in the waiting hall, number of low priority customers in the orbit and the status of the server, $\Phi^{(i_1)}\mathbf{e}$ gives the probability of i_1 item in the inventory in the steady state. Hence η_I is given by

$$
\eta_I = \sum_{i_1=1}^{S} i_1 \Phi^{(i_1)}\mathbf{e}.
$$

4.2 Expected Reorder Rate

Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s+1$ to s. This may occur in the following two cases:

- an arrival of a low priority customer,
- an arrival of a orbit customer,
- the server completes a service for the high priority customer,
- any one of the s items fails when the server is busy,

- any one of the $(s+1)$ items fails when the server is idle.

Hence we get

$$\begin{aligned}\eta_R &= ((M-i_4)\lambda_2 + (s+1)\gamma + \theta_{i_4})\sum_{i_4=0}^{N_2}\phi^{(s+1,0,0,i_4)} + \\ &\quad ((M-i_3-i_4)\lambda_2 + s\gamma + \theta_{i_4})\sum_{i_2=1}^{k-1}\sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2}\phi^{(s+1,0,0,i_4)} + \\ &\qquad ((M-i_3-i_4)\lambda_2 + s\gamma + \theta_{i_4} + k\mu)\sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2}\phi^{(s+1,k,i_3,i_4)}.\end{aligned}$$

4.3 Expected Perishable Rate

Since $\Phi^{(i_1)}$ is the steady state probability vector for inventory level, the expected perishable rate η_P is given by

$$\eta_P = \sum_{i_1=1}^{S}\sum_{i_4=0}^{N_2} i_1\gamma\phi^{(i_1,0,0,i_4)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2}(i_1-1)\gamma\phi^{(i_1,i_2,i_3,i_4)}.$$

4.4 Expected Number of High Priority Customers in the Waiting Area

Let Γ_1 denote the expected number of high priority customers in the steady state. Since $\phi^{(i_1,i_2,i_3,i_4)}$ is a vector of probabilities with the inventory level is i_1, the server status is i_2, the number of high priority customers in the waiting area is i_3 and the number of low priority customers in the orbit is i_4, the expected number of high priority customers Γ_1 in the steady state is given by

$$\Gamma_1 = \sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2} i_3\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2} i_3\phi^{(i_1,i_2,i_3,i_4)}.$$

4.5 Expected Waiting Time

Let η_W denote the expected waiting time of the high priority customers in the waiting area. Then by Little's formula

$$\eta_W = \frac{\Gamma_1}{\eta_{AH}},$$

where Γ_1 is the expected number of high priority customers in the waiting area and the effective arrival rate of the high priority customer (Ross [14]), η_{AH} is given by

$$\begin{aligned}\eta_{AH} &= \sum_{i_3=0}^{N_1-1}\sum_{i_4=0}^{N_2}(M-i_3-i_4)\lambda_1\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_4=0}^{N_2}(M-i_4)\lambda_1\phi^{(i_1,0,0,i_4)} \\ &\quad + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1-1}\sum_{i_4=0}^{N_2}(M-i_3-i_4)\lambda_1\phi^{(i_1,i_2,i_3,i_4)}.\end{aligned}$$

4.6 Expected Number of Low Priority Customers in the Orbit

Let Γ_2 denote the expected number of low priority customers in the steady state. Since $\phi^{(i_1,i_2,i_3,i_4)}$ is a vector of probabilities with the inventory level is i_1, the server status is i_2, the number of high priority customers in the waiting area is i_3 and the number of low priority customers in the orbit is i_4, the expected number of low priority customers Γ_2 in the steady state is given by

$$\Gamma_2 = \sum_{i_3=0}^{N_1}\sum_{i_4=1}^{N_2} i_4\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_4=1}^{N_2} i_4\phi^{(i_1,0,0,i_4)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=1}^{N_2} i_4\phi^{(i_1,i_2,i_3,i_4)}.$$

4.7 Expected Number of High Priority Customers Lost

Let η_{HL} denote the expected number of high priority customers lost to the system in the steady state. Any arriving high priority customer finds the waiting area is full and leaves the system without getting service. These customers are considered to be lost. Thus we obtain

$$\eta_{HL} = \sum_{i_4=0}^{N_2}(M-N_1-i_4)\lambda_1\phi^{(0,0,N_1,i_4)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_4=0}^{N_2}(M-N_1-i_4)\lambda_1\phi^{(i_1,i_2,N_1,i_4)}.$$

4.8 Expected Number of Low Priority Customers Lost

Let η_{LL} denote the expected number of low priority customers lost to the system in the steady state. Any arriving low priority customer finds the orbit size is full when either inventory level is zero or server is busy while only one item in the inventory and leaves the system not entering the orbit. These customers are considered to be lost. Thus we obtain

$$\eta_{LL} = \sum_{i_3=0}^{N_1}(M-i_3-N_2)\lambda_2\phi^{(0,0,i_3,N_2)} + \sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}(M-i_3-N_2)\lambda_2\phi^{(1,i_2,i_3,N_2)}.$$

4.9 Probability that Server is Busy with a High Priority Customer

Let η_{PH} denote the probability that server is busy with a high priority customer is given by

$$\eta_{PH} = \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=0}^{N_2}\phi^{(i_1,i_2,i_3,i_4)}.$$

4.10 Probability that Server is Idle

Let η_{PI} denote the probability that server is idle is given by

$$\eta_{PI} = \sum_{i_3=0}^{N_1}\sum_{i_4=0}^{N_2}\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_4=0}^{N_2}\phi^{(i_1,0,0,i_4)}.$$

4.11 The Overall Rate of Retrials

Let η_{OR} denote the overall rate of retrials in the steady state. Then we have

$$\eta_{OR} = \sum_{i_3=0}^{N_1}\sum_{i_4=1}^{N_2} \theta_{i_4}\phi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_4=1}^{N_2} \theta_{i_4}\phi^{(i_1,0,0,i_4)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=1}^{N_2} \theta_{i_4}\phi^{(i_1,i_2,i_3,i_4)}.$$

4.12 The Successful Retrial Rate

Let η_{SR} denote the successful retrial rate in the steady state . Then

$$\eta_{SR} = \sum_{i_1=1}^{S}\sum_{i_4=1}^{N_2} \theta_{i_4}\phi^{(i_1,0,0,i_4)} + \sum_{i_1=2}^{S}\sum_{i_2=1}^{k}\sum_{i_3=1}^{N_1}\sum_{i_4=1}^{N_2} \theta_{i_4}\phi^{(i_1,i_2,i_3,i_4)}.$$

4.13 The Fraction of Successful Rate of Retrial

Let η_{FR} denote the fraction of successful retrial rate in the steady state . Then

$$\eta_{FR} = \frac{\eta_{SR}}{\eta_{OR}}.$$

5. Expected Total Cost Rate

To compute the total expected cost per unit time (total expected cost rate), the following costs, are considered.

c_h : The inventory carrying cost per unit item per unit time
c_s : Setup cost per order
c_p : Perishable cost per unit item per unit time
c_{bh} : Cost per high priority customer lost
c_{bl} : Cost per low priority customer lost
c_{wh} : Waiting cost of a high priority customer per unit time
c_{wl} : Waiting cost of a low priority customer per unit time

The long run total expected cost rate is given by

$$TC(S, s, N_1, N_2) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_h\eta_w + c_l\Gamma_2 + c_{hl}\eta_{HL} + c_{ll}\eta_{LL},$$

where $\eta' s$ are as given in Sections 4.1-4.8.

6. Concluding Remarks

In this article, we analyzed a continuous review stochastic retrial queueing-inventory system with two types of customers, (s, S) replenishment policy and finite population. Retrial is introduced for low priority customers only. The lead times of reorder, service times and the retrial demand time points form independent exponential distributions. The model is analyzed within the framework of Markov processes. Joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level is obtained in the steady state. Various system performance measures are derived and the long-run total expected cost rate is derived. The authors are working in the direction of MAP (Markovian arrival process) arrival for the two types of customers and service times follow PH-distributions.

Acknowledgements. The author wishes to thank the anonymous referee who has provided valuable suggestions for improving the presentation of this paper.

References

[1] Artalejo, J.R., Krishnamoorthy, A. and Lopez-Herrero, M.J. *Numerical analysis of (s,S) inventory system with repeated attempts.* Annals of Operations Research, 2006, **141**, No 1, pp. 67-83

[2] Berman, O. and Kim, E. *Stochastic models for inventory management at service facility.* Stochastic Models, 1999, **15**, No 4, pp. 695-718

[3] Berman, O. and Kim, E. *Dynamic inventory strategies for profit maximization in a service facility with stochastic service.* Mathematical Methods of Operations Research, 2004, **60**, No 3, pp. 497-521

[4] Berman, O. and Kim, E. *Dynamic order replenishment policy in internet-based supply chains.* Mathematical Methods of Operations Research, 2001, **53**, No 3, pp. 371-390

[5] Berman, O. and Sapna, K.P. *Inventory management at service facility for systems with arbitrarily distributed service times.* Communications in Statistics. Stochastic Models, 2001, **16**, No 3, pp. 343-360

[6] Berman, O. and Sapna, K.P. *Optimal control of service for facilities holding inventory.* Computers and Operations Research, 2001, **28**, No 5, pp. 429-441

[7] Choi, B.D. and Chang, Y. *Single server retrial queues with priority calls.* Mathematical and Computer Modelling, 2001, **30**, pp. 7-32

[8] Jeganathan, K., Anbazhagan, N. and Kathiresan, J. *A retrial inventory system with non-preemptive priority service.* International Journal of Information and Management Sciences, 2013, **24**, pp. 57-77

[9] Krishnamoorthy, A. and Jose, K.P. *Comparison of inventory systems with service, positive lead-time, loss, and retrial of customers.* Journal of Applied Mathematics and Stochastic Analysis, 2007, **2007**, pp. 1-23.

[10] Krishnamoorhy, A., Narayanan, V.C., Deepak, T.G. and Vineetha, P. *Control policies for inventory with service time.* Stochastic Analysis and Applications, 2006, **24**, No 4, pp. 889-899

[11] Ning, Z. and Zhaotong, L. *A queueing-inventory system with two classes of customers.* International journal of Production Economics, 2011, **129**, pp. 225-231

[12] Schwarz, M. and Daduna, H. *Queueing systems with inventory management with random lead times and with backordering.* Mathematical Methods of Oprations Research, 2006, **64**, No 3, pp. 383-414

[13] Schwarz, M., Sauer, C., Daduna, H., Kulik, R. and Szekli, R. *M/M/1 queueing systems with inventory.* Queueing Systems Theory and Applications, 2006, **54**, No 1, pp. 55-78

[14] Ross, S. M. *Introduction to Probability Models.* 2000, Harcourt Asia PTE Ltd, Singapore

[15] Ushakumari, P.V. *On (s,S) inventory system with random lead time and repeated demands.* Journal of Applied Mathematics and Stochastic Analysis, 2006, **2006**, pp. 1-22