http://mmg.tversu.ru ISSN 2311-1275

MMG *Mathematical Modelling and Geometry*

Volume 2, No 1, p. 27 – 32 (2014)

Finite dynamical uniaxial strain of nonlinear elastic solids: an exact solution

Konstantin M. Zingerman[1,a]

[1] Faculty of Applied Mathematics and Cybernetics, Tver State University, Sadovyi per. 35, Tver, Russia

e-mail: [a] Konstantin.Zingerman@tversu.ru

Received 7 February 2014, in final form 22 March 2014. Published 25 march 2014.

Abstract. The uniaxial dynamical equation of nonlinear elasticity is solved for the case of finite strains. A partial exact solution is found for the case when the material properties are described by the Blatz–Ko strain energy function.

Keywords: nonlinear dynamics, finite strains, one-dimensional problem, exact solution

MSC numbers: 74B20, 74H05

The work is partially supported by the Russian Foundation for Basic Research (grant 14–08–01191)

1. Introduction

The subject of this paper is the solving of the system of dynamical equations of nonlinear elasticity for uniaxial large deformations. It should be noted that that a number of similar problems were solved earlier. For example, solutions for nonlinear waves (solitons) in solids are obtained in [2, 8, 10]. Also, some solutions for incompressible materials are known [3, 4].

An exact solution of one of the dynamical problems of nonlinear elasticity is obtained in [12], where the material properties are described by the two-constant potential (a particular case of the Murnaghan potential [7]). The solution is self-similar and is found using the similarity method [11]. In this paper, we use the same approach for the Blatz–Ko material.

2. Statement of the problem for the Blatz–Ko material

The problem is formulated in the frame of initial (undeformed) configuration. The system of equations is written in tensorial form and consists of the following equations:

The equation of motion

$$\overset{0}{\nabla} \cdot \boldsymbol{\mathcal{P}} + \rho_0 \, \mathbf{f} = \rho_0 \, \mathbf{a} \,, \tag{1}$$

where ρ_0 is the initial material density, $\mathbf{f}$ is the vector of body forces (hereinafter this vector is neglected), $\mathbf{a} = \dfrac{\partial^2 \mathbf{u}}{\partial t^2}$ is the acceleration vector, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\mathcal{P}}$ is the first Piola stress tensor [7],

$$\boldsymbol{\mathcal{P}} = (1 + \Delta) \boldsymbol{\Psi}^{* \, -1} \cdot \boldsymbol{\sigma}, \tag{2}$$

and $\boldsymbol{\sigma}$ is the Cauchy stress tensor;

Kinematic equations

$$\mathbf{G} = \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^{*}, \quad \mathbf{F} = \boldsymbol{\Psi}^{*} \cdot \boldsymbol{\Psi}, \quad (1 + \Delta) = \det \boldsymbol{\Psi} \,, \quad \boldsymbol{\Psi} = \mathbf{I} + \overset{0}{\nabla} \mathbf{u}. \tag{3}$$

Here $\mathbf{G}$ is the Cauchy–Green deformation tensor, $\mathbf{F}$ is the Finger strain tensor, $\boldsymbol{\Psi}$ is the deformation gradient, Δ is the relative volume variation, and $\mathbf{I}$ is the identity tensor.

The Blatz–Ko potential has a form [1]

$$\mathrm{W} = \frac{1}{2} \mu \beta \left[I_1 + \frac{1}{\alpha} (I_3^{-\alpha} - 1) - 3 \right] + \frac{1}{2} \mu (1 - \beta) \left[I_2 I_3^{-1} + \frac{1}{\alpha} (I_3^{\alpha} - 1) - 3 \right].$$

, where $I_k = I_k(\mathbf{G})$ are invariants of the Cauchy–Green deformation tensor $\mathbf{G}$.

An exact solution is obtained for the particular case of this potential with $\beta = 0$ and $\alpha = 1/2$. The constitutive equations for this case may be written as

$$\boldsymbol{\sigma} = \mu [I_3(\mathbf{G})]^{-3/2} \left\{ \left[[I_3(\mathbf{G})]^{3/2} - I_2(\mathbf{G}) \right] \mathbf{I} + I_1(\mathbf{G}) \mathbf{F} - \mathbf{F}^2 \right\}. \tag{4}$$

3. The solution of the problem for one-dimensional case

We investigate the one-dimensional motion in the direction of x-axis when $\mathbf{u} = u(x)\mathbf{e}_1$. In this case

$$\mathbf{F} = \mathbf{G} = (1+u_x)^2\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3, \tag{5}$$

$$I_1(\mathbf{G}) = (1+u_x)^2 + 2, \quad I_2(\mathbf{G}) = 2(1+u_x)^2 + 1, \quad I_3(\mathbf{G}) = (1+u_x)^2. \tag{6}$$

Substitution of Eqs. (5) and (6) into Eq. (4), taking into account the relation (2), yields:

$$\mathcal{P}_{11} = \sigma_{11} = \mu\left[1-(1+u_x)^{-3}\right]. \tag{7}$$

Next, substituting this expression into the equation of motion (1), we obtain

$$\mu\frac{\partial}{\partial x}\left[1-(1+u_x)^{-3}\right] = \rho_0 u_{tt}. \tag{8}$$

This equation may be written in the form

$$u_{tt} = [f(u_x)]_x, \tag{9}$$

where $f(u_x) = k^2\left[1-(1+u_x)^{-3}\right]$, $k = \sqrt{\mu/\rho_0}$.

In order to solve equation 9, we shall use the new independent variable $y = x/t$ and seek the solution in the form $u(x,t) = t\,U(x/t) = tU(y)$. As a result, we have the equation

$$\frac{U''(y)\left\{y^2[1+U'(y)]^4 - 3k^2\right\}}{t[1+U'(y)]^4} = 0. \tag{10}$$

This equation can be decomposed into the two equations:

$$U''(y) = 0 \tag{11}$$

and

$$y^2[1+U'(y)]^4 - 3k^2 = 0. \tag{12}$$

The solution of Eq. (11) is

$$U(y) = C_1 y + C_2,$$

where C_1 and C_2 are constants. After the replacement $y = x/t$ and substitution of this solution into the expression $u(x,t) = t\,U(x/t)$, one finds

$$u(x,t) = C_1 x + C_2 t.$$

This solution describes a homogeneous deformation coupled with the uniform motion. Eq. (12) can be solved with respect to $U'(y)$:

$$U'(y) = -1 \pm \sqrt[4]{3}\sqrt{k/y}. \tag{13}$$

where we assume $k > 0$.

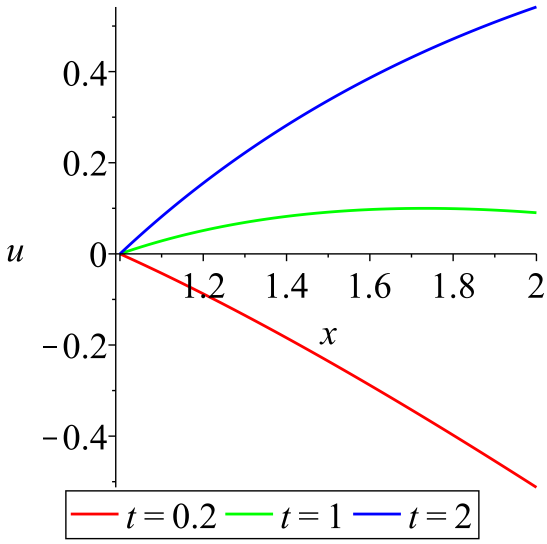

Figure 1: The distribution of u along x-axis at different moments in time.

Integrating Eq. (13) we have

$$U(y) = -y \pm 2\sqrt[4]{3}\sqrt{ky} + C. \tag{14}$$

If we replace y by x/t in Eq. (14) and substitute the result into the expression $u(x,t) = t\,U(x/t)$, we obtain the final formula

$$u(x,t) = -x \pm 2\sqrt[4]{3}\sqrt{ktx} + Ct. \tag{15}$$

A boundary condition should be specified in order to find the constant C. In particular, one can write the boundary condition in the form $u(a,t) = 0$, where a is a given point. Note that the solution above is finite at $t = 0$ and tends to infinity as $t \to \infty$. It can be shown that the true stresses tend to infinity as $x \to 0$.

The plots of the solution are shown in Figs. 1 and 2 for the case $k = 1$, $u(1,t) = 0$, and with the positive sign before the second term in the formula (15). The distribution of the displacement u along x-axis at different moments in time is presented in Fig. 1, and the distribution of the true stress σ_{11} along x-axis at different moments in time is shown in Fig. 2.

4. Conclusion

We obtain an exact analytical solution of the one-dimensional dynamical problem of nonlinear elasticity for the Blatz–Ko material under finite strains. This solution may be used in testing of numerical solutions [6] for dynamical elasticity problems under finite strains.

Acknowledgements. The author is grateful to Prof. V.A. Levin for useful discussions.

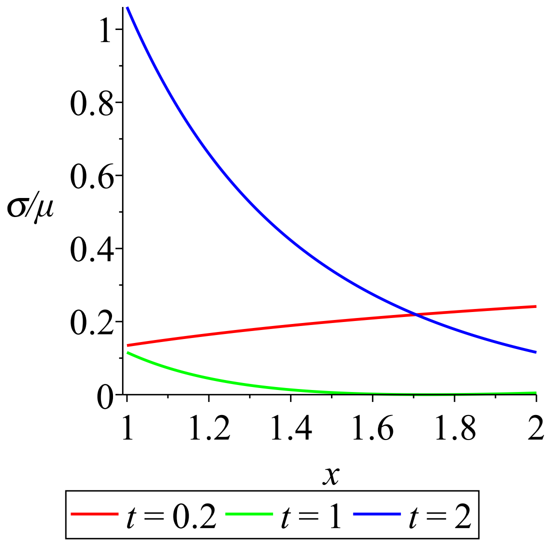

Figure 2: The distribution of true stress σ_{11} along x-axis at different moments in time.

References

[1] Blatz P.J. and Ko W.L. *Applications of finite elasticity theory to deformation of rubbery materials.* Trans. Soc. Rheol. 1962, **6**, pp. 223–251

[2] Dai H.-H. and Huo Y. *Solitary shock waves and other travelling waves in a general compressible hyperelastic rod.* Proc. Roy. Soc. London A 2000, **456**, pp. 331–363

[3] Destrade M. *Finite-amplitude inhomogeneous plane waves in a deformed Mooney-Rivlin material.* Quarterly Journal of Mechanics and Applied Mathematics 2000, **53**, pp. 343-–361

[4] Hill J.M. and Wegner J.L. *New families of exact nonlinear waves for the neo-Hookean finite elastic solid.* Mathematics and Mechanics of Solids 2004, **9**, pp. 81–95

[5] Knowles J.K. *Large amplitude oscillations of a tube of incompressible elastic material.* Q. Appl. Math. 1960. **18**, pp. 71–77

[6] Levin V.A. and Vershinin A.V. *Non-stationary plane problem of the successive origination of stress concentrators in a loaded body. Finite deformations and their superposition.* Communications in Numerical Methods in Engineering 2008, **24**, pp. 2229–2239

[7] Lurie A.I. *Nonlinear theory of elasticity.* 1990, Amsterdam: North-Holland

[8] Miroshnichenko A.E., Vasiliev A.A., Dmitriev S.V. and Shigenari T. *Inelastic three-soliton collisions in a weakly discrete sine-Gordon system.* Nonlinearity 2000, **13**, No. 3. pp. 837–848

[9] Murnaghan F.D. *Finite deformation of an elastic solid.* 1951, New York: Wiley, 140pp.

[10] Samsonov A.M. *Strain solitons in solids and how to construct them.* 2001, Chapman & Hall/CRC Press, 227pp.

[11] Sedov L.I. *Similarity and Dimensional Methods in Mechanics, Tenth Edition.* 1993, CRC Press, 496pp.

[12] Zingerman K.M. *Exact solution of one dynamic problem in the elasticity theory under finite strains.* 2013, Vestnik TvGU. Seriya: Prikladnaya matematika [Herald of Tver State University. Series: Applied Mathematics], No. 3(30), pp. 19–25 (in Russian)