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On isoclinic left Bol three-webs with the trivial core

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Abstract. We find equations for a multidimensional isoclinic left Bol three-web whose core is assumed to be trivial (that is, to be isotopic to an Abelian group).

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MSC numbers: 53A60

1. Introduction

It is known [1, 2, 3] that any left Bol three-web (a web $B_l \equiv B_l(r,r,r)$) induces a locally symmetric structure on the base of the first foliation. This structure is produced by a local smooth quasigroup called the core of a web B_l. The notion of core has been first introduced by V.D. Belousov in [4]. It is known, that the core of a web B_l is not, in general, isotopic to the coordinate quasigroup of the web. In particular, the core of a web B_l can be isotopic to an Abelian group (that is, be trivial). At the same time, a web B_l may not be parallelizable, so that its coordinate quasigroup is not, in general, isotopic to an Abelian group. In [4] it is shown that a symmetric structure produced by a core (trivial or not) is locally flat. See also [5] for more detail on locally flat structure produced by a Bol web. In [6] a left Bol three-web with the trivial core is denoted by B_l^0 and its structure equations are obtained there.

In this paper, we consider isoclinic left Bol three-webs B_l with trivial core (isoclinic webs B_l^0). It is known that each isoclinic middle Bol web is algebraizable, that is, it is equivalent to a Grassmann web defined by three hypersurfaces X_α, $\alpha = 1,2,3$, belonging to a hypercubic on the Grassmann manifold of straight lines of the projective space P^{r+1}. Points of the hypersurfaces X_1, X_2, and X_3 determines the bundles of straight lines and the bundles represent leaves of the first, second, and the third foliations of the web B_l. It is shown in Section 4 that any isoclinic left Bol three-web is algebraizable one of a special type (Theorem 1). We find the curvature tensor $\tilde{R}^i_{jkl}$ of symmetric connection defined by the core of isoclinic left Bol web B_l on its first foliation (Proposition 1). The web B_l^0 is characterized by the condition $\tilde{R}^i_{jkl} = 0$ and by taking into account this property we find the structure equations of the isoclinic three web B_l^0 (Proposition 2). By integrating of these structure equations we obtain the equations of the coordinate quasigroup of the web B_l^0 (Theorem 2).

2. The core of a left Bol web

Definition 1. *Three smooth foliations λ_1, λ_2, and λ_3 given on $2r$-dimensional differentiable manifold M is called a three-web $W(r,r,r)$ if any two of these foliations are in the general position and their leaves are of the dimension r.*

Following [2] we specify the foliations of the web $W(r,r,r)$ in local coordinates on the manifold M by the equations

$$\lambda_1: \ x = const, \quad \lambda_2: \ y = const, \quad \lambda_3: \ z = f(x,y) = const,$$

where $x = (x^1,...,x^r)$, $x \in X$, $y = (y^1,...,y^r)$, $y \in Y$, $z = (z^1,...,z^r)$, $z \in Z$, and the function $f = (f^1,...,f^r)$ is smooth and satisfies the conditions $\left|\dfrac{\partial f}{\partial x}\right| \neq 0$, $\left|\dfrac{\partial f}{\partial y}\right| \neq 0$ at each point of the manifold $\mathcal{M}$.

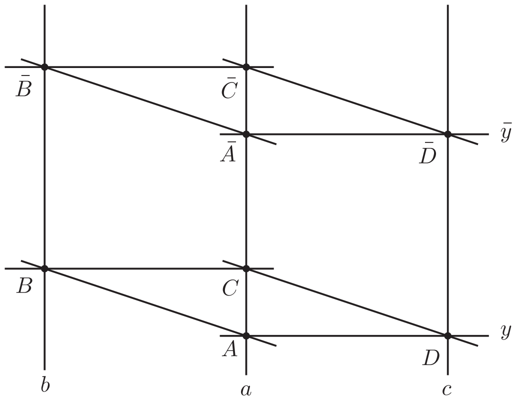

Figure 1: A left Bol configuration.

The equation $z = f(x, y)$ is called *the equation of the three-web* $W(r, r, r)$. It connects the parameters x, y, and z of leaves passing through a point and belonging to the first, second, and third foliations respectively. So, the equation defines a smooth local quasigroup

$$(\cdot): \; X \times Y \to Z, \quad z = f(x, y) \equiv x \cdot y, \tag{1}$$

which is called *coordinate quasigroup* of the three-web $W(r, r, r)$. The variables x, y, and z admit transformations of the type

$$\tilde{x} = \alpha(x), \quad \tilde{y} = \beta(y), \quad \tilde{z} = \gamma(z),$$

where α, β, and γ are local diffeomorphisms and the triple (α, β, γ) is called *an isotopic transformation* [2]. By this transformation the equation (1) takes the form

$$\tilde{z} = \tilde{f}(\tilde{x}, \tilde{y}) = \gamma \circ f(\alpha^{-1}(\tilde{x}), \beta^{-1}(\tilde{y})),$$

which determines another three-web $\tilde{W}(r, r, r)$, being equivalent to the three-web $W(r, r, r)$ [2].

In [2] basic classes of three-webs are specified, including the left Bol web $B_l \equiv B_l(r, r, r)$. The three-web B_l is characterizes by the condition that its coordinate quasigroup is a left Bol loop. Recall that a loop is a left Bol loop if it satisfies the left Bol identity

$$(u \circ (v \circ u)) \circ w = u \circ (v \circ (u \circ w)), \quad (\circ \text{ is the loop operation}).$$

This identity corresponds to the left Bol configuration (B_l) (see Figure1 where leaves of the first, second, and third foliations of the web are pictured as usually by vertical, horizontal, and inclined straight lines respectively). Let us remind its standard scheme.

Let a and b are two arbitrary sufficiently close to one another vertical leaves, and y an arbitrary horizontal leaves. The latter intersects the leave a in some point

A. There is a unique inclined leave through A that intersects the leave b in some point B. There is a horizontal leave through B that intersects the vertical leave a in some point C. The unique inclined leave, passing through the point C, intersect the leave y in some point D. Analogously, for another horizontal leave $\bar{y}$, we obtain the new points $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. One says that the configuration (B_l) is closed [2] if the points D and $\bar{D}$ lie in one and the same vertical leave (in Figure1 it is denoted by c).

Definition 2. *A three-web is called a left Bol web and denoted by B_l if all sufficiently small configurations (B_l) are closed on it.*

The identity

$$f(a, f^{-1}(b, f(a, y))) = f(a * b, y), \quad a, b \in X, \quad y \in Y,$$

where $c = a * b$ is a local quasigroup, corresponds to the condition of the closing of configurations (B_l) on the three-web $W(r, r, r)$.

Definition 3. *A quasigroup $(*) : \ X \times X \to X$, $c = a * b$, defined by the rule*

$$c \ = \ {}^{-1}f(f(a, f^{-1}(b, f(a, y))), y) \ \ \forall y \in Y,$$

is called the core of a three-web B_l.

It is known that the core $c = a * b$ is idempotent $(a * a = a)$, left invertible $(a * (a * b) = b)$, and left distributive $(a * (b * c) = (a * b) * (a * c))$, so that it is isotopic to left Bol loop [2]). The core of a three-web is not, in general, isotopic to coordinate quasigroup of the web.

3. A locally symmetric connection produced by the core of a left Bol web

The core of a left Bol three-web $B_l \equiv B_l(r, r, r)$ defines a family of smooth functions S_a on the base X of the first foliation of the web, such that $S_a(b) = a * b$ for any $a \in X$ and $b \in U_a \subset X$, where U_a is a sufficiently small neighborhood of the point a (see Figure1). The functions S_a are local symmetries and the manifold $\{X, S_a\}$ is a symmetric space. Structure equations of the connection, denoted by $\tilde{\Gamma}$, can be obtained as follows.

Let $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ are basis of differential 1-forms on the manifold $\mathcal{M}$ which is support for the web $W(r, r, r)$; hereafter $i, j, k, ... = \overline{1, r}$. The forms define leaves of the web

$$\lambda_1 : \ \underset{1}{\omega}^i = 0, \quad \lambda_2 : \ \underset{2}{\omega}^i = 0, \quad \lambda_3 : \ \underset{3}{\omega}^i \equiv \underset{1}{\omega}^i + \underset{2}{\omega}^i = 0 \tag{2}$$

and satisfy the structure equations [2]

$$d\underset{1}{\omega}^i = \underset{1}{\omega}^j \wedge \omega^i_j + a^i_{jk}\underset{1}{\omega}^j \wedge \underset{1}{\omega}^k, \quad d\underset{2}{\omega}^i = \underset{2}{\omega}^j \wedge \omega^i_j - a^i_{jk}\underset{2}{\omega}^j \wedge \underset{2}{\omega}^k, \tag{3}$$

$$d\omega_j^i = \omega_j^k \wedge \omega_k^i + b^i_{jkl} \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l. \tag{4}$$

The quantities a^i_{jk} and b^i_{jkl} are, respectively, the torsion tensor and the curvature tensor of the three-web $W(r,r,r)$. The forms

$$(\underset{1}{\omega}^i,\ \underset{2}{\omega}^i) \quad \text{and} \quad \begin{pmatrix} \omega^i_j & 0 \\ 0 & \omega^i_j \end{pmatrix}$$

define an affine connection on the manifold $\mathcal{M}$, which is called the canonical (or Chern) connection [2], and its torsion and curvature components coincide with quantities a^i_{jk} and, respectively, b^i_{jkl} for the three-web $W(r,r,r)$.

In accordance with [2], the three-webs $B_l \equiv B_l(r,r,r)$ are characterize by the conditions

$$b^i_{(jk)l} = 0. \tag{5}$$

We suppose that

$$\tilde{\omega}^i_j = \omega^i_j + a^i_{jk} \underset{1}{\omega}^k \tag{6}$$

and reduce equations (3) and (4) to the form

$$\begin{aligned} d\underset{1}{\omega}^i &= \underset{1}{\omega}^j \wedge \tilde{\omega}^i_j, \\ d\underset{2}{\omega}^i &= \underset{2}{\omega}^j \wedge \tilde{\omega}^i_j - a^i_{jk} \underset{2}{\omega}^j \wedge \underset{3}{\omega}^k, \\ d\tilde{\omega}^i_j &= \tilde{\omega}^k_j \wedge \tilde{\omega}^i_k + \tilde{R}^i_{jkl} \underset{1}{\omega}^k \wedge \underset{1}{\omega}^l, \end{aligned} \tag{7}$$

where forms $\{\underset{1}{\omega}^i,\ \tilde{\omega}^i_j\}$ determine the connection $\tilde{\Gamma}$. Its curvature tensor has the form

$$\tilde{R}^i_{jkl} = \frac{1}{4}(b^i_{klj} - 2a^i_{mj} a^m_{kl}) \tag{8}$$

and is covariantly constant, that is,

$$\tilde{\nabla} \tilde{R}^i_{jkl} \equiv d\tilde{R}^i_{jkl} - \tilde{R}^i_{mkl}\tilde{\omega}^m_j - \tilde{R}^i_{jml}\tilde{\omega}^m_k - \tilde{R}^i_{jkm}\tilde{\omega}^m_l + \tilde{R}^m_{jkl}\tilde{\omega}^i_m = 0.$$

We can show that

$$\tilde{\nabla} a^i_{jk} \equiv da^i_{jk} - a^i_{mk}\tilde{\omega}^m_j - a^i_{jm}\tilde{\omega}^m_k + a^m_{jk}\tilde{\omega}^i_m = \frac{1}{2} b^i_{jkl}(\underset{3}{\omega}^l + \underset{2}{\omega}^l), \tag{9}$$

$$\begin{aligned} \tilde{\nabla} b^i_{jkl} \equiv db^i_{jkl} - b^i_{mkl}\tilde{\omega}^m_j - b^i_{jml}\tilde{\omega}^m_k - b^i_{jkm}\tilde{\omega}^m_l + b^m_{jkl}\tilde{\omega}^i_m &= \\ = (a^i_{pl} b^p_{jkm} + a^p_{jk} b^i_{plm})(\underset{3}{\omega}^m + \underset{2}{\omega}^m)&, \end{aligned} \tag{10}$$

and the torsion and curvature tensors for the web B_l are tied by formulae

$$a^i_{pl} b^p_{jkm} - a^i_{pm} b^p_{jkl} + b^i_{plm} a^p_{jk} - b^i_{pml} a^p_{jk} + b^i_{jkp} a^p_{lm} = 0.$$

On account of (8), one can write these formulae as

$$a^i_{pl}\tilde{R}^p_{mjk} - a^i_{pm}\tilde{R}^p_{ljk} + \tilde{R}^i_{mpl} a^p_{jk} - \tilde{R}^i_{lpm} a^p_{jk} + \tilde{R}^i_{pjk} a^p_{lm} + \frac{1}{2} a^i_{pq} a^p_{jk} a^q_{lm} = 0. \tag{11}$$

4. Isoclinic left Bol three-webs

In this section we show that any isoclinic Bol three-web is grassmannizable one of a special type. Recall [2] that an isoclinic three-webs is characterized by the following conditions for the torsion tensor:

$$a^i_{jk} = a_{[j}\delta^i_{k]}. \tag{12}$$

In addition to equations (12), an Grassmannizable web should be satisfied the conditions for the curvature tensor:

$$b^i_{jkl} = b^1_{jk}\delta^i_l + b^2_{lj}\delta^i_k + b^3_{kl}\delta^i_j, \tag{13}$$

where b^1_{jk}, b^2_{jk} and b^3_{jk} are symmetric tensors [2].

Let a web B_l be isoclinic, so that conditions (5) and (12) are satisfied. Writing equations (9) and taking account (12), we obtain

$$\tilde{\nabla} a_j\delta^i_k - \tilde{\nabla} a_k\delta^i_j = b^i_{jkl}(\underset{3}{\omega}^l + \underset{2}{\omega}^l) \tag{14}$$

where $\tilde{\nabla} a_j \equiv da_j - a_m\tilde{\omega}^m_j$. According to [2], the equations

$$\nabla a_j \equiv da_j - a_m\omega^m_j = p_{jl}\underset{1}{\omega}^l + q_{jl}\underset{2}{\omega}^l, \tag{15}$$

and

$$b^i_{[jk]l} = q_{l[j}\delta^i_{k]}, \quad b^i_{[j|l|k]} = p_{l[j}\delta^i_{k]} \tag{16}$$

are satisfied for an arbitrary isoclinic web; in addition, if $r > 2$ then

$$b^i_{[jk]l} = q_{l[j}\delta^i_{k]}, \quad b^i_{[j|l|k]} = p_{l[j}\delta^i_{k]}. \tag{17}$$

In virtue of (12), it follows from (6) that $\tilde{\omega}^i_j = \omega^i_j + \dfrac{1}{2}(a_j\delta^i_k - a_k\delta^i_j)\underset{1}{\omega}^k$, so that, by taking into account (15), we obtain

$$\tilde{\nabla} a_j = \nabla a_j - \frac{1}{2}a_m(a_j\delta^m_l - a_l\delta^m_j)\underset{1}{\omega}^l = p_{jl}\underset{1}{\omega}^l + q_{jl}\underset{2}{\omega}^l. \tag{18}$$

Taking the last equations into account, the relations

$$q_{jl} = 2p_{jl}, \tag{19}$$

$$b^i_{jkl} = p_{jl}\delta^i_k - p_{kl}\delta^i_j \tag{20}$$

follow from (14), and equations (16) are identities because of (5), (17), and (18). By comparing (19) with (13) we find

$$b^1_{jk} = 0, \quad b^2_{jk} = p_{jk}, \quad b^3_{jk} = -p_{jk}, \tag{21}$$

so that the isoclinic web B_l is grassmannizable, and its curvature tensor has the form (20).

Conversely, a grassmannizable three-web with the curvature tensor of the form (ref20) the isoclinic web B_l, because the conditions (ref12) hold for grassmannizable webs, and conditions (ref12) follow from (20). So, one has

Theorem 1. *An isoclinic left Bol three-web (and only a web of this type) is a grassmannizable web with the curvature tensor of the form (20).*

Let us point out a geometric sense of conditions (21). Recall [2], that a three-web is said to be grassmannizable if it is equivalent to a Grassmannian web that produced on the Grassmann manifold $G(1, r+1)$ of straight lines of the projective space P^{r+1} by three hypersurfaces X_α, $\alpha = 1, 2, 3$ belonging to the same hypercubic. Points of the hypersurfaces X_1, X_2, and X_3 define the bundles of straight lines that represent leaves of, respectively, the first, second, and third foliations of the web B_l. The quantities b^α_{jk} are coefficients of the asymptotic quadratic form of the hypersurface X_α. The conditions (21) mean that the hypersurface X_1 is a hyperplane, and the hypersurfaces X_1 and X_2 belong to the same hyperquadric Q of the space P^{r+1} [2].

Thus, isoclinic left Bol three-webs make up a web class whose webs are equivalent to Grassmannian webs defined by a hyperplane and a hyperquadric.

Let us write formulae for covariant differentials of the tensors a_i and p_{ij} of the isoclinic web B_l. Because of (19), equations (18) for the tensor a_i take the form

$$\tilde{\nabla} a_i = p_{ij}(\underset{3}{\omega}^j + \underset{2}{\omega}^j). \tag{22}$$

Now we can use equations (10) for the curvature tensor of the web B_l. Taking into account formulae (12) and (20), they have become

$$\tilde{\nabla} p_{jl}\delta^i_k - \tilde{\nabla} p_{kl}\delta^i_j = \frac{1}{2}(-a_l p_{jm}\delta^i_k + a_l p_{km}\delta^i_j - a_j p_{lm}\delta^i_k + a_k p_{lm}\delta^i_j)(\underset{3}{\omega}^m + \underset{2}{\omega}^m).$$

Let us contract these equalities over the indices i and j, and take into account that $\sum_i \delta^i_i = r$ and $r > 2$. The result is

$$\tilde{\nabla} p_{kl} = -\frac{1}{2}(a_l p_{km} + a_k p_{lm})(\underset{3}{\omega}^m + \underset{2}{\omega}^m). \tag{23}$$

Substituting expressions (12) into formulae (8) and (20), we find the form of the tensor $\tilde{R}^i_{jkl}$ for the isoclinic web B_l:

$$\tilde{R}^i_{jkl} = \frac{1}{4}((p_{jk} + \frac{1}{2}a_j a_k)\delta^i_l - (p_{jl} + \frac{1}{2}a_j a_l)\delta^i_k). \tag{24}$$

So, one has

Proposition 1. *For the isoclinic web B_l, the curvature tensor $\tilde{R}^i_{jkl}$ of the symmetric connection $\tilde{\Gamma}$, defined on the base of the first foliation, has the form (24) where the tensors a_j and p_{jk} satisfy equations (22) and (23).*

Notice that above arguments are similar those stated in [2] for the middle Bol webs (B_m webs). According to [2], the isoclinic web B_m is interpreted in the space P^{r+1} as a Grassmann web which is produced by the hyperplane X_3 and the hyperquadric Q containing hypersurface X_1 and X_2. As well as in [2], we suppose that $r > 2$, since it is known that any four-dimensional Bol web is grassmannizable. In [5] four-dimensional webs B_m are classified by the form of the quadric Q and its mutual location with the plane X_3. A Bol web is parallelizable if $r = 1$ [2].

5. An isoclinic web B_l with the trivial core

Let us consider the isoclinic web B_l with the locally flat symmetric connection $\tilde{\Gamma}$, so that the corresponding curvature tensor is zero:

$$\tilde{R}^i_{jkl} = 0. \tag{25}$$

Such a connection is denoted by $\tilde{\Gamma}^0$ in [6], and the corresponding three-web is denoted by B^0_l. It is proved in [6] that the core $c = a * b$ of a web B^0_l (and only a web of this type) is trivial (isotopic to an Abelian group).

In order to find the structure equations of the isoclinic three-web B^0_l we use the structure equations of an three-web B^0_l obtained in [6]. The forms $\tilde{\omega}^i_j$ is shown in [6] to be reduce to zero everywhere on the manifold $\mathcal{M}$ supporting a three-web B^0_l, and the structure equations are obtained from (7) and have the form

$$\begin{aligned} &d\underset{1}{\omega}^i = 0,\\ &d\underset{2}{\omega}^i = -a^i_{jk}\underset{2}{\omega}^j \wedge \underset{3}{\omega}^k. \end{aligned}$$

Taking into account conditions (12), which characterize isoclinic webs, we write these structure equations in the form

$$\begin{aligned} &d\underset{1}{\omega}^i = 0,\\ &d\underset{2}{\omega}^i = -\frac{1}{2}(a_j\delta^i_k - a_k\delta^i_j)\underset{2}{\omega}^j \wedge \underset{3}{\omega}^k. \end{aligned} \tag{26}$$

The equalities

$$(p_{jk} + \frac{1}{2}a_j a_k)\delta^i_l - (p_{jl} + \frac{1}{2}a_j a_l)\delta^i_k = 0$$

follow from (24) and (25). Contracting these equalities over indices i and l, and taking into account conditions $\sum\limits_i \delta^i_i = r$ and $r \geq 2$ we obtain

$$p_{jk} = -\frac{1}{2}a_j a_k. \tag{27}$$

The conditions $\tilde{\omega}^i_j = 0$ implies $\tilde{\nabla} a_i = da_i$, so that equations (22) on account of (27) have the form

$$da_i = -\frac{1}{2}a_i a_j(\underset{3}{\omega}^j + \underset{2}{\omega}^j). \tag{28}$$

Next, from (11) on account of (25) we obtain the equalities

$$a^i_{pq}a^p_{jk}a^q_{ml} = 0.$$

By taking into account (12) one can directly verify that these equalities are identically true. Equations (23) are also identically true because of (27) and (28).

Let us point out explicitly the curvature tensor b^i_{jkl} for an isoclinic web B^0_l. Substituting expressions (27) into equalities (20) we obtain

$$b^i_{jkl} = -\frac{1}{2}(a_j\delta^i_k - a_k\delta^i_j)a_l. \tag{29}$$

So, one has

Proposition 2. *The structure equations of an isoclinic three-web B_l^0 (and only of a web of this type) can be reduced to the form (26) where the covector a_i is satisfied equations (28). The torsion tensor and the curvature tensor of an isoclinic web are determined by formulae (12) and (29) respectively.*

Now we find equations for the coordinate quasigroup of an isoclinic web B_l^0 by integrating of structure equations (26). First, let us consider equations (28). Supposing

$$\Theta = \frac{1}{2}a_j(\underset{1}{\omega}^j + 2\underset{2}{\omega}^j), \tag{30}$$

we write their in the form

$$da_i = -a_i\Theta. \tag{31}$$

In this case

$$d\Theta = 0. \tag{32}$$

On account of (30) we write structure equations (26) in the form

$$\begin{aligned} &d\underset{1}{\omega}^i = 0, \\ &d\underset{2}{\omega}^i = \underset{2}{\omega}^i \wedge \Theta + \frac{1}{2}\underset{1}{\omega}^i \wedge \left(\Theta - \frac{1}{2}a_j\underset{1}{\omega}^j\right). \end{aligned} \tag{33}$$

By consecutive integrating of equations (32) and (31) we obtain

$$\Theta = -d\phi, \tag{34}$$

$$a_i = C_i e^{\phi}, \tag{35}$$

where ϕ is a function, and $C_i = const$. It follows from (35), (30), and (33) that a three-web B_l^0 is parallelizable, if $C_i = 0\ \forall\, i = \overline{1, r}$. Hence we further consider webs that have at least one non-zero constant C_i.

Substituting (34) and (35) into system (33) we write it in the form

$$\begin{aligned} &d\underset{1}{\omega}^i = 0, \\ &d\underset{2}{\omega}^i = -\underset{2}{\omega}^i \wedge d\phi - \frac{1}{2}\underset{1}{\omega}^i \wedge (d\phi + \frac{1}{2}e^{\phi}C_j\underset{1}{\omega}^j). \end{aligned} \tag{36}$$

By integrating equations (36) we find the basis forms

$$\begin{aligned} &\underset{1}{\omega}^i = du^i, \\ &\underset{2}{\omega}^i = \frac{1}{4}e^{\phi}C_j u^j du^i - \frac{1}{2}du^i + e^{\phi}dv^i, \end{aligned} \tag{37}$$

where $(u^i,\, v^i)$ are local coordinates on the manifold $\mathcal{M}$. Substituting (34), (35),and (35) into (30), we obtain the equation

$$de^{-2\phi} = \frac{1}{2}C_k u^k d(C_j u^j) + 2d(C_j v^j).$$

Integration of this equation yields

$$e^{-2\phi} = \frac{1}{4}(C_j u^j)^2 + 2C_j v^j + \phi_0, \tag{38}$$

where $\phi_0 = const$.

Let us find equations for the leaves of the web under consideration. Writing equations (2), (37) and taking into account (38), we obtain

$$\begin{aligned} \lambda_1 : &\quad du^i = 0, \\ \lambda_2 : &\quad C_j u^j du^i - ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} du^i + 4dv^i = 0, \\ \lambda_3 : &\quad C_j u^j du^i + ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} du^i + 4dv^i = 0. \end{aligned} \tag{39}$$

By integrating the first equations in system (39), we find

$$u^i = x^i, \tag{40}$$

where x^i are parameters of the leaves of the first foliation.

Now let us consider the second series of equations (39). Since covector C_i does not equal zero, we contract these equations with C_i and obtain the equation

$$C_j u^j d(C_i u^i) - ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} d(C_i u^i) + 4d(C_i v^i) = 0,$$

which is equivalent to

$$\frac{1}{2}((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{-1/2} d((C_i u^i)^2 + 8(C_i v^i) + 4\phi_0) = d(C_i u^i).$$

Integrating the first equation and denoting the constant of integration by y^1, we have

$$((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} = C_i u^i + y^1. \tag{41}$$

Taking into account (41) we write the remaining $r-1$ equations, which determine the foliation λ_2, in the form

$$C_j u^j du^i - (C_j u^j + y^1) du^i + 4dv^i = 0, \quad i = \overline{2, r}.$$

Integrating these equations and denoting the constants of integration by y^i, we find

$$-y^1 u^i + 4v^i = y^i, \quad i = \overline{2, r}. \tag{42}$$

Thus, the leaves of the second foliation of the web are determined by equations (41) and (42), where y^i are parameters of the leaves, and $i = \overline{1, r}$.

We find the equations for the second and third leaves analogously. In order to do this we write the third series of the equations of system (39) in the form

$$\begin{aligned} &C_j u^j d(C_k u^k) + ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} d(C_k u^k) + 4d(C_k v^k) = 0, \\ &C_j u^j du^i + ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} du^i + 4dv^i = 0, \quad i = \overline{2, r}. \end{aligned}$$

This system is equivalent to the following one:

$$\frac{1}{2}((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{-1/2} d((C_j u^j)^2 + 8(C_j v^j) + 4\phi_0) = -d(C_j u^j),$$
$$C_j u^j du^i + ((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} du^i + 4dv^i = 0, \quad i = \overline{2, r}.$$

The integration of this system yields equations for the leaves of the third foliation in the form

$$\begin{aligned} &((C_j u^j)^2 + 8C_j v^j + 4\phi_0)^{1/2} = -C_j u^j + z^1, \\ &z^1 u^i + 4v^i = z^i, \quad i = \overline{2, r}, \end{aligned} \tag{43}$$

where z^i are parameters of the leaves, $i = \overline{1, r}$. Eliminating the local coordinates from equations (40) – (43) and supposing that

$$\bar{x}^1 = 2C_j x^j, \tag{44}$$

we obtain the equations for the isoclinic web B_l^0:

$$\begin{cases} z^1 = \bar{x}^1 + y^1, \\ z^i = (\bar{x}^1 + 2y^1)x^i + y^i, \quad i = \overline{2, r}. \end{cases} \tag{45}$$

On the other hand, these equations determine the coordinate quasigroup of the considered web.

Let us find the torsion and curvature tensors of the three-web determined by equations (45). In these equations $(\bar{x}^1, x^2, ..., x^r, y^1, y^2, ..., y^r)$ are parameters of the leaves of the first and second foliations, and on the other hand, they can be considered as local coordinates on the manifold supporting the three-web (see ([2])). In the local coordinates $C_1 = \frac{1}{2}$, $C_j = 0$, $j = \overline{2, r}$ (see (44) and hence we obtain from (38) and (35), because of (40) and (41) respectively, that

$$e^{\phi} = \frac{4}{\bar{x}^1 + 2y^1},$$

$$a_1 = \frac{2}{\bar{x}^1 + 2y^1}, \quad a_i = 0, \quad i = \overline{2, r}. \tag{46}$$

On account of the last expressions, we find from (12) that the considered web has the following non-zero components of the torsion tensor:

$$a^i_{1i} = \frac{1}{\bar{x}^1 + 2y^1}, \quad i = \overline{2, r} \quad \text{(no summation over } i). \tag{47}$$

Now let us find the components b^i_{jkl} of the curvature tensor of the considered web. From (29) and because of (26), we have

$$b^i_{jk1} = -\frac{1}{\bar{x}^1 + 2y^1}(a_j \delta^i_k - a_k \delta^i_j), \quad b^i_{jkl} = 0, \quad l = \overline{2, r}.$$

Further, for $k=1$ and $k=\overline{2,r}$ we obtain respectively

$$b^i_{j11}=-\frac{1}{\bar{x}^1+2y^1}(a_j\delta^i_1-a_1\delta^i_j),\quad b^i_{jk1}=-\frac{1}{\bar{x}^1+2y^1}a_j\delta^i_k,\quad k=\overline{2,r},$$

so that only the following components can have non-zero values:

$$b^i_{j11}=\frac{2}{(\bar{x}^1+2y^1)^2}\delta^i_j=-b^i_{1j1},\quad j=\overline{2,r}.$$

It implies that components of the type

$$b^i_{i11}=\frac{2}{(\bar{x}^1+2y^1)^2}\quad \text{(no summation over } i) \tag{48}$$

are non-zero for any $i=\overline{2,r}$.

The calculation by formula (24) shows that all the components of the curvature tensor of a locally symmetric connection $\tilde{\Gamma}$ induced by the considered web are equal to zero, that is, the connection $\tilde{\Gamma}$ is locally flat, just as it should be for a web with the trivial core.

Thus, we have the following theorem:

Theorem 2. *The equations for multidimensional isoclinic left Bol three-webs with the trivial core can be reduced to the form (45) in some local coordinates, and the torsion and curvature tensors of the web have non-zero components of the form (47) and (48) respectively.*

In particular for $r=2$, we obtain a four dimensional web B^0_l, for which a four dimensional middle Bol web B_m is uniquely defined. Recall [2] that the web B_m, directly tied with a web B_l, is determined by the inverse quasigroup $x={}^{-1}f(z,y)$ of the coordinate quasigroup $z=f(x,y)$ of the three-web B_l. In this case, the vertical leaves of the web B_l become the inclined leaves of the web B_m, inclined leaves of the web B_l become the vertical leaves of the web B_m, and horizontal leaves of the web B_l become the horizontal leaves of the web B_m.

It is already said above that four dimensional webs B_m was classified in [7] on account of their projective interpretation in P^3. In this classification, we point out the type of the four dimensional webs B_m tied with the considered web B^0_l. Let us find the equations of the left inverse coordinate quasigroup (45) for $r=2$ and rename the variables: $x\leftrightarrow z$, $y\rightarrow -y$. As a result, we obtain the equations of the four dimensional web B_m in the form

$$z^1=x^1+y^1,\quad z^2=\frac{x^2+y^2}{x^1-y^1}.$$

This web is shown in [5] to be a web of the parabolic type Π_2. The quadric Q in P^3 is a cone, and the plane X_3 is tangent to the cone along a rectilinear generator [7].

6. Conclusion

In the article we describe the class of isoclinic left Bol three-webs with the trivial core. It is shown that any isoclinic web B_l is an algebraizable web of the special type (Theorem 1). We find the form of the curvature tensor $\tilde{R}^i_{jkl}$ of the symmetric connection $\tilde{\Gamma}$ determined by the core of an isoclinic web B_l on the base of its first foliation (Proposition 1). If the connection $\tilde{\Gamma}$ is locally flat ($\tilde{R}^i_{jkl} = 0$), then the core of the web B^0_l defining the connection is trivial, that is, it is isotopic to an Abelian group. We find the structure equations of an isoclinic three-web B^0_l (Proposition 2). By integrating of of these equations we obtain the equations of the coordinate quasigroup of a multidimensional three-web B^0_l (Theorem 2). Since the grassmannizable three-web web are interpreted by a specified way on the Grassmann manifold $G(1, r + 1)$ [2], a promising problem is to construct a projective model of the class of the isoclinic webs B^0_l.

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