http://mmg.tversu.ru ISSN 2311-1275

Mathematical Modelling and Geometry

Volume 1, No 3, p. 38 – 48 (2013)

Optimal control of predator-prey model with distributed delay

E.A. Andreeva[1,a], **I.S. Mazurova**[1,b]

[1] Department of Mathematics, Tver State University, Sadovyi per. 35, Tver, Russia
e-mail: [a] andreeva.tvgu@yandex.ru, [b] irinasmazurova@gmail.com

Received 1 October 2013, in final form 2 December 2013. Published 3 December 2013.

Abstract. The purpose of this research is to describe the Lotka – Volterra biological model using the system of integro-differential equations. The necessary conditions of optimality obtained with the help of maximum principle are analyzed here. The optimal control for different types of minimizing functionals is determined with the help of necessary conditions of optimality and the multipoint boundary-value problem is formulated. Numerical methods and the algorithm are developed to find the optimal process. The obtained numerical results correspond to the theoretical conclusions of maximum principle

Keywords: model Lotka-Volterra type, integro-differential equations, maximum principle, numerical methods
MSC numbers: 92D25, 65K10

The second authors are supported by the CCAS, (grant SS - 5264.2012.1).

Mathematical models of the biological populations are described by means of ordinary differential equations, differential equations with delay, integro-differential equations and also models based on discrete equations on extremal principles and with the help of neural networks.

In this article we analyze the mathematical model of interaction between m different types of prey and k different types of predators described by integro-differential equations, that is generalization of model, described in [1].

In this paper we consider a special case when k types of predators compete for m types of prey. By $x_i(t), i = 1, m$ we denote the population of the prey species, by $y_j(t), j = 1, k$ the population of k types of predators. By the real nonnegative constant parameters R_i we denote the constant amount of prey population which is inaccessible to predators. The time evolution in this model can be described by the Volterra-type integro-differential equations

$$\dot{x}_i(t) = x_i(t)\left(e_i - \sum_{l=1}^{m} a_{il}x_l(t)\right) - \sum_{j=1}^{n} b_{ji}y_j(t)(x_i(t) - R_i) - u_i(t),\ i = \overline{1,m} \quad (1)$$

$$\begin{aligned} \dot{y_j}(t) &= y_j(t)\left(-\alpha_j - \sum_{l=1}^{n} c_{jl}y_l(t) + \sum_{l=1}^{m} d_{jl}(x_l(t) - R_l)\right) + \\ &+ y_j(t)\sum_{l=1}^{m}\gamma_{jl}\int_{t-r}^{t} F_{jl}(t-\tau)(x_l(\tau) - R_l)d\tau - v_j(t),\ j = \overline{1,k} \end{aligned} \quad (2)$$

with initial dates

$$x_i(0) = x_i^0, x_i(\theta) = \varphi_i(\theta), y_j(0) = y_j^0, \theta \in [-r, 0],\ i = \overline{1,m}, j = \overline{1,k} \quad (3)$$

where $\varphi_i(\theta)$ are given continuous functions, $e_i, \alpha_j, a_{il}, b_{jl}, c_{jl}, d_{jl}, \gamma_{jl}$ are real positive constants which characterize the intersection of population.

The functions $F_{jl}(t-\tau)$ describe the influence of the past on the present evolution of the predator species of catching rates.

The control functions $u_i(t)$ are rates of catching prey i, $v_j(t)$ are rates of catching predator j which satisfy the following restrictions

$$0 \le u_i(t) \le u_{imax}, 0 \le v_j(t) \le v_{jmax},\ t \in [0, T], i = \overline{1,m}, j = \overline{1,n} \quad (4)$$

where constants u_{imax}, v_{jmax} are the given functions maximal rate of catching.

For example, depending on the technology of catching, the following constraints could be imposed on the control functions:

$$\sum_{i=1}^{m}\gamma_i u_i(t) \le B,\ 0 \le u_i(t) \le u_{imax},$$

$$\sum_{j=1}^{k}\rho_j v_j(t) \le A,\ 0 \le v_j(t) \le v_{jmax},$$

or

$$u_i(t) = \alpha_i x_i(t) u, \ \ v_j(t) = \beta_j y_j(t) v(t).$$

The goal of control is to minimize (maximize) the cost functional

$$J(u,v) = \int_0^T f_0(t,x,y,u,v)dt + \Phi(x(T),y(T)), \tag{5}$$

on the set of admissible processes (1)-(3) and control constraints (4).

In many practical problems the criteria of the optimal catching are to receive maximum profit for a company or to keep populations on the prescribed level at the end of catching or on the whole interval $[0,T]$.

We have used the following functional:

$$\begin{aligned} J_1(u,v) \quad &= \int_0^T (\textstyle\sum_{i=1}^m [\rho_i(t,x_i(t))u_i(t) - d_i(t,u_i(t))] + \\ &+ \textstyle\sum_{j=1}^k [\tilde{\rho}_j(t,y_j(t))v_j(t) - \tilde{d}_j(t,v_j(t))])e^{-\lambda t}dt \end{aligned} \tag{6}$$

where the functions $\rho_i(t,x_i), \tilde{\rho}_j(t,y_j)$ are the prices on market and $d_i(t,u_i), \tilde{d}_j(t,v_j)$ are the prices of technologies, λ is discount parameter.

Terminal functional

$$J_2(u,v) = \sum_{i=1}^m M_i(x_i(T) - A_i)^2 + \sum_{j=1}^k N_j(y_j(T) - B_j)^2 \tag{7}$$

is responsible for the preservation of the populations on the given level $x_i(T) = A_i, i = \overline{1,m}, y_j(T) = B_j, j = \overline{1,k}$ at the end of the process.

The final constraints $x_i(T) \geq A_i, i = \overline{1,m}, y_j(T) \geq B_j, j = \overline{1,k}$ can be taken in consideration by penalty functions

$$J_3(u,v) = \sum_{i=1}^m M_i \left(max(A_i - x_i(T), 0)^{2n}\right) + \sum_{j=1}^k N_j(max(B_j - y_j(T), 0)^{2n}) \tag{8}$$

where $M_i, i = \overline{1,m}, N_j, j = \overline{1,k}$, are the positive penalty coefficients.

In case of state constraints on the interval $x_i(t) \geq A_i, i = \overline{1,m}, y_j(t) \geq B_j, j = \overline{1,k}, t \in [0,T]$ one can use special maximum principle for the optimal control problem with state constraints or the method of penalty functions

$$J_4(u,v) = \int_0^T \left(\sum_{i=1}^m M_i max(A_i - x_i(t), 0)^2 + \sum_{j=1}^k N_j max(B_j - y_j(t), 0)^2 \right) dt \tag{9}$$

The multicriteria problem of optimisation and the properties of Pareto domains are analyzed in paper.

The necessary conditions of optimality in the form of Pontryagin maximum principle were used to determine optimal control $\overline{u}(t), \overline{v}(t), t \in [0,T]$.

According to the maximum principle for the problem (1)-(5) optimal control $\overline{u}(t), \overline{v}(t), t \in [0,T]$ can be found from the expression:

$$\max_{0\le\omega_i\le u_{max},0\le\psi_i\le v_{max}} [-\lambda_0 f_0(t,\overline{x}(t),\overline{y}(t),\omega,\psi) - \sum_{i=1}^{m} p_i(t)\omega_i - \sum_{j=1}^{k} r_j(t)\psi_j] =$$
$$-\lambda_0 f_0(t,\overline{x}(t),\overline{y}(t),\overline{u}(t),\overline{v}(t)) - \sum_{i=1}^{m} p_i(t)\overline{u}_i(t) - \sum_{j=1}^{k} r_j(t)\overline{v}_j(t), \quad (10)$$

where adjoint functions are the solutions to the system of integro-differential equations

$$\begin{aligned} \dot{p}_i(t) &= -p_i(t)\left(e_i - \sum_{l=1}^{m} a_{il}x_l(t)\right) + \sum_{l=1}^{m} p_l(t)a_{li}(t)x_l(t) + p_i(t)\sum_{j=1}^{k} b_{ji}y_j(t) \\ &- \sum_{j=1}^{k} r_j(t)y_j(t)d_{ji} - \sum_{j=1}^{k} r_j(t)y_j(t)\gamma_{ji}\int_t^{t+r} F_{ji}(\tau - t)d\tau, i = \overline{1,m} \end{aligned} \quad (11)$$

$$\begin{aligned} \dot{r}_j(t) &= r_j(t)\alpha_j + r_j(t)\sum_{l=1}^{k} c_{jl}(t)y_l(t) + \sum_{l=1}^{m} p_l(t)b_{lj}(x_l(t) - R_l) \\ &+ \sum_{l=1}^{k} r_l(t)y_l(t)c_{lj} - r_l(t)\sum_{l=1}^{m} d_{jl}(t)(x_l(t) - R_l) \\ &- r_j(t)\sum_{l=1}^{m}\gamma_{jl}\int_{t-r}^{t} F_{jl}(t-\tau)(x_l(\tau) - R_l)d\tau, j = \overline{1,k} \end{aligned} \quad (12)$$

with transversality conditions at the end of the integration interval

$$\begin{aligned} p_i(T) &= -\lambda_0\frac{\partial J_2(x(T),y(T))}{\partial x_i}, p_i(t) \equiv 0, \texttt{if } t > T, i = \overline{1,m}, \\ r_j(T) &= -\lambda_0\frac{\partial J_2(x(T),y(T))}{\partial y_j}, r_j(t) \equiv 0, \texttt{if } t > T, j = \overline{1,k}. \end{aligned} \quad (13)$$

If the functions $d_i(t,u_i), \tilde{d}_j(t,v_j)$ are linear on u_i, v_j, then optimal control depends on signs of switching functions $\psi_i(t), \varphi_j(t)$:

$$\psi_i(t) = \lambda_0 e^{-\lambda t}\left(\rho_i(t,x_i) - d_i(t)\right) - p_i(t), i = \overline{1,m},$$

$$\varphi_j(t) = \lambda_0 e^{-\lambda t}\left(\tilde{\rho}_j(t, y_j) - \tilde{d}_j(t)\right) - r_j(t), j = \overline{1,k},$$

In this case optimal control satisfies the following expressions

$$\overline{u}_i(t) = \begin{cases} u_{imax}, & \text{if } \psi_i(t) > 0 \\ 0, & \text{if } \psi_i(t) < 0 \end{cases}$$

$$\overline{v}_j(t) = \begin{cases} v_{jmax}, & \text{if } \varphi_j(t) > 0 \\ 0, & \text{if } \varphi_j(t) < 0 \end{cases} \tag{14}$$

In the case $\psi_i(t) = 0$ or $\varphi_j(t) = 0$ we have singular arcs maximum principle could not be applied and Kelly conditions must be investigated.

For the case $F_{jl}(t-\tau) = \eta_{jl} e^{\varepsilon_{jl}(t-\tau)}$ one can introduce the functions $z_{jl}(t) = \int\limits_{t-r}^{t} F_{jl}(t-\tau)(x_l(\tau) - R_l)d\tau$, then the system of integro-differential equations modifies to the system of differential equations with constant delay:

$$\dot{x}_i(t) = x_i(t)(e_i - \sum_{l=1}^{m} a_{il}x_l(t)) - \sum_{j=1}^{n} b_{ji}y_j(t)(x_i(t) - R_i) - u_i(t), i = \overline{1,m} \tag{15}$$

$$\begin{aligned} \dot{y}_j(t) &= y_j(t)\left(-\alpha_j - \sum_{l=1}^{n} c_{jl}y_l(t) + \sum_{l=1}^{m} d_{jl}(x_l(t) - R_l)\right) + \\ &+ y_j(t)\sum_{l=1}^{m} \gamma_{jl}z_{jl}(t) - v_j(t), j = \overline{1,n} \end{aligned} \tag{16}$$

$$\dot{z}_{jl}(t) = \eta_{jl}(x_i(t) - R_i) - F_{jl}(r)(x_l(t-r) - R_l) + \varepsilon_{jl}z_{jl}, j = \overline{1,k}, l = \overline{1,m} \tag{17}$$

Optimal control satisfies the maximum principle (10) and adjoint functions $p_i(t), r_j(t), s_{ji}(t), i = \overline{1,m}, j = \overline{1,k}$ are the solution to the system of differential equations with deviated argument

$$\begin{aligned} \dot{p}_i(t) &= \lambda_0 \frac{\partial f_0(t, x(t), y(t), u(t), v(t))}{\partial x_i} - p_i(t)\left(e_i - \sum_{l=1}^{m} a_{il}x_l(t)\right) \\ &+ \sum_{l=1}^{m} p_l(t)a_{li}(t)x_l(t) + p_i(t)\sum_{j=1}^{k} b_{ji}y_j(t) - \sum_{j=1}^{k} r_j(t)y_j(t)d_{ji} \\ &- \sum_{j=1}^{k} s_{ji}(t)\eta_{ji} - s_{ji}(t+r)e^{\varepsilon_{ji}(r)}, i = \overline{1,m} \end{aligned} \tag{18}$$

$$\begin{aligned}
\dot{r}_j(t) &= \lambda_0 \frac{\partial f_0(t, x(t), y(t), u(t), v(t))}{\partial y_j} + r_j(t)\alpha_j + r_j(t)\sum_{l=1}^{k} c_{jl}(t)y_l(t) \\
&+ \sum_{l=1}^{m} p_l(t)b_{lj}(x_l(t) - R_l) + \sum_{l=1}^{k} r_l(t)y_l(t)c_{lj} - r_l(t)\sum_{l=1}^{m} d_{jl}(t)(x_l(t) - R_l) \\
&- r_j(t)\sum_{l=1}^{m} \gamma_{jl} z_{jl}(t), j = \overline{1,k} \qquad (19)
\end{aligned}$$

$$\dot{s}_{ji}(t) = -r_j(t)y_j(t)\gamma_{ji} - \varepsilon_{ji}s_{ji}(t), j = \overline{1,k}, i = \overline{1,m} \qquad (20)$$

and transversality conditions

$$\begin{aligned}
p_i(T) &= -\lambda_0 \frac{\partial J_2(x(T), y(T))}{\partial x_i}, p_i(t) \equiv 0, \ \texttt{if}\ t > T, i = \overline{1,m}, \\
r_j(T) &= -\lambda_0 \frac{\partial J_2(x(T), y(T))}{\partial y_j}, r_j(t) \equiv 0, \ \texttt{if}\ t > T, j = \overline{1,k}, \\
s_{jl}(T) &= 0, s_{jl}(t) \equiv 0, \ \texttt{if}\ t > T, j = \overline{1,k}, l = \overline{1,m}.
\end{aligned}$$

If the delay parameter r is small then $x_l(t-r) = x_l(t) - r\dot{x}_l(t) + O(r)$, and the system (18)-(20) can be transferred to the system of ordinary differential equations

$$\dot{x}_i(t) = x_i(t)(e_i - \sum_{l=1}^{m} a_{il}x_l(t)) - \sum_{j=1}^{k} b_{ji}y_j(t)(x_i(t) - R_i) - u_i(t), i = \overline{1,m} \qquad (21)$$

$$\begin{aligned}
\dot{y_j}(t) &= y_j(t)(-\alpha_j - \sum_{l=1}^{n} c_{jl}y_l(t) + \sum_{l=1}^{m} d_{jl}(x_l(t) - R_l)) + \\
&+ y_j(t)\sum_{l=1}^{m} \gamma_{jl}z_{jl}(t) - v_j(t), j = \overline{1,n} \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\dot{z_{jl}}(t) &= \eta_{jl}(x_i(t) - R_i) - F_{jl}(r)(x_l(t) - rx_i(t)(e_i - \sum_{l=1}^{m} a_{il}x_l(t)) \\
&+ r\sum_{j=1}^{k} b_{ji}y_j(t)(x_i(t) - R_i) + ru_i(t) - R_i) + \varepsilon_{jl}z_{jl}, j = \overline{1,k}, i = \overline{1,m} \qquad (23)
\end{aligned}$$

and optimal control satisfies maximum principle

$$\max_{0\le\omega_i\le u_{max},0\le\psi_i\le v_{max}}[-\lambda_0 f_0(t,\overline{x}(t),\overline{y}(t),\omega,\psi)-\sum_{i=1}^{m}p_i(t)\omega_i-\sum_{j=1}^{k}r_j(t)\psi_j]=$$

$$-\lambda_0 f_0(t,\overline{x}(t),\overline{y}(t),\overline{u}(t),\overline{v}(t))-\sum_{i=1}^{m}(p_i(t)+\sum_{i=1}^{k}s_{ji}(t)\eta_{jl}e^{\varepsilon_{jl}r}r)\overline{u}_i(t)-\sum_{j=1}^{k}r_j(t)\overline{v}_j(t), \quad (24)$$

where adjoint functions $p_i(t), r_j(t), s_{ji}(t), i=\overline{1,m}, j=\overline{1,k}$ are the solution to the system of ordinary differential equations

$$\begin{aligned}
\dot{p}_i(t) &= \lambda_0\frac{\partial f_0(t,x(t),y(t),u(t),v(t))}{\partial x_i}-p_i(t)(e_i-\sum_{l=1}^{m}a_{il}x_l(t))+\sum_{l=1}^{m}p_l(t)a_{li}(t)x_l(t)\\
&+ p_i(t)\sum_{j=1}^{k}b_{ji}y_j(t)-\sum_{j=1}^{k}r_j(t)y_j(t)d_{ji}-\sum_{j=1}^{k}s_{ji}(t)\eta_{ji}-s_{ji}(t)e^{\varepsilon_{ji}(r)}\\
&- \sum_{j=1}^{k}s_{ji}(t)\eta_{ji}e^{\varepsilon_{ji}(r)}r(e_i-\sum_{l=1}^{m}a_{il}x_l(t))+r\sum_{j=1}^{k}\sum_{l=1}^{m}s_{jl}(t)\eta_{jl}e^{\varepsilon_{jl}(r)}a_{li}x_l(t))\\
&+ r\sum_{j=1}^{k}s_{ji}(t)\eta_{ji}e^{\varepsilon_{ji}(r)}\sum_{l=1}^{k}b_{li}y_l(t), i=\overline{1,m} \qquad (25)
\end{aligned}$$

$$\begin{aligned}
\dot{r}_j(t) &= \lambda_0\frac{\partial f_0(t,x(t),y(t),u(t),v(t))}{\partial y_j}+r_j(t)\alpha_j+r_j(t)\sum_{l=1}^{k}c_{jl}(t)y_l(t)\\
&+ \sum_{l=1}^{m}p_l(t)b_{lj}(x_l(t)-R_l)+\sum_{l=1}^{k}r_l(t)y_l(t)c_{lj}-r_l(t)\sum_{l=1}^{m}d_{jl}(t)(x_l(t)-R_l)\\
&- r_j(t)\sum_{l=1}^{m}\gamma_{jl}z_{jl}(t)+r\sum_{l=1}^{m}s_{jl}(t)\eta_{jl}e^{\varepsilon_{jl}(r)}b_{lj}(x_l(t)-R_l), j=\overline{1,k} \qquad (26)
\end{aligned}$$

$$\dot{s}_{ji}(t)=-r_j(t)y_j(t)\gamma_{ji}-\varepsilon_{ji}s_{ji}(t), j=\overline{1,k}, i=\overline{1,m} \qquad (27)$$

with transversality conditions

$$\begin{aligned}
p_i(T) &= -\lambda_0\frac{\partial J_2(x(T),y(T))}{\partial x_i},\ i=\overline{1,m},\\
r_j(T) &= -\lambda_0\frac{\partial J_2(x(T),y(T))}{\partial y_j},\ j=\overline{1,k},\\
s_{jl}(T) &= 0,\ j=\overline{1,k}, l=\overline{1,m}.
\end{aligned}$$

Consider the case when $F_{jl}(t-\tau) = \begin{cases} \eta_{jl}, & \tau \in [t-r,t] \\ 0, & \tau \notin [t-r,t] \end{cases}$ and determine functions $z_{jl}(t) = \int\limits_{t-r}^{t} F_{jl}(t-\tau)(x_l(\tau)-R_l)d\tau$. Then the system of integro-differential equations (1)-(3) is reduced to the system of differential equations with delay

$$\dot{x}_i(t) = x_i(t)(e_i - \sum_{l=1}^{m} a_{il}x_l(t)) - \sum_{j=1}^{k} b_{ji}y_j(t)(x_i(t)-R_i) - u_i(t), i = \overline{1,m} \quad (28)$$

$$\begin{aligned} \dot{y_j}(t) &= y_j(t)(-\alpha_j - \sum_{l=1}^{n} c_{jl}y_l(t) + \sum_{l=1}^{m} d_{jl}(x_l(t)-R_l)) + \\ &+ y_j(t)\sum_{l=1}^{m}\gamma_{jl}z_{jl}(t) - v_j(t), j = \overline{1,n} \end{aligned} \quad (29)$$

$$\dot{z_{jl}}(t) = \eta_{jl}(x_l(t) - x_l(t-r)), j = \overline{1,k}, l = \overline{1,m} \quad (30)$$

Optimal control for the problem (28)-(30) with cost functional (5) satisfies the maximum principle (10) and adjoint functions $p_i(t), r_j(t), s_{ji}(t), i = \overline{1,m}, j = \overline{1,k}$ are the solution to the system of differential equations with deviated argument:

$$\begin{aligned} \dot{p}_i(t) &= \lambda_0 \frac{\partial f_0(t,x(t),y(t),u(t),v(t))}{\partial x_i} - p_i(t)(e_i - \sum_{l=1}^{m} a_{il}x_l(t)) \\ &+ \sum_{l=1}^{m} p_l(t)a_{li}(t)x_l(t) + p_i(t)\sum_{j=1}^{k} b_{ji}y_j(t) - \sum_{j=1}^{k} r_j(t)y_j(t)d_{ji} \\ &- \eta_{ji}\sum_{j=1}^{k} s_{ji}(t) - s_{ji}(t+r), i = \overline{1,m} \end{aligned} \quad (31)$$

$$\begin{aligned} \dot{r}_j(t) &= \lambda_0 \frac{\partial f_0(t,x(t),y(t),u(t),v(t))}{\partial y_j} + r_j(t)\alpha_j + r_j(t)\sum_{l=1}^{k} c_{jl}(t)y_l(t) \\ &+ \sum_{l=1}^{m} p_l(t)b_{lj}(x_l(t)-R_l) + \sum_{l=1}^{k} r_l(t)y_l(t)c_{lj} \\ &- r_l(t)\sum_{l=1}^{m} d_{jl}(t)(x_l(t)-R_l) - r_j(t)\sum_{l=1}^{m}\gamma_{jl}z_{jl}(t), j = \overline{1,k} \end{aligned} \quad (32)$$

$$\dot{s}_{ji}(t) = -r_j(t)y_j(t)\gamma_{ji}, j = \overline{1,k}, i = \overline{1,m} \tag{33}$$

and transversality condition

$$\begin{aligned}
p_i(T) &= -\lambda_0 \frac{\partial J_2(x(T), y(T))}{\partial x_i}, p_i(t) \equiv 0, \ \texttt{if}\ t > T, i = \overline{1,m}, \\
r_j(T) &= -\lambda_0 \frac{\partial J_2(x(T), y(T))}{\partial y_j}, r_j(t) \equiv 0, \ \texttt{if}\ t > T, j = \overline{1,k}, \\
s_{jl}(T) &= 0, s_{jl}(t) \equiv 0, \ \texttt{if}\ t > T, j = \overline{1,k}, l = \overline{1,m}.
\end{aligned}$$

Mathematical modeling has become an essential tool in studying predator-prey interactions. The system considered in this paper represents one of many possible generalizations of classical Lotka-Volterra approach to multispecies interactions. Most researchers study the convergence to equilibrium or global stability. In this article we use necessary conditions of optimality in the form of principle maximum which allows us to transform the optimal control problem to the multipoint boundary-value problem. The special cases of this problem have been concerned in articles [2], [3].

Numerical simulations of the models are represented with different parameter values. Figures 1 - 2 demonstrate the influence of the delay parameter on the evolution of the functions $x_1(t), y_1(t)$ on time interval $t \in [0, T]$ without control.

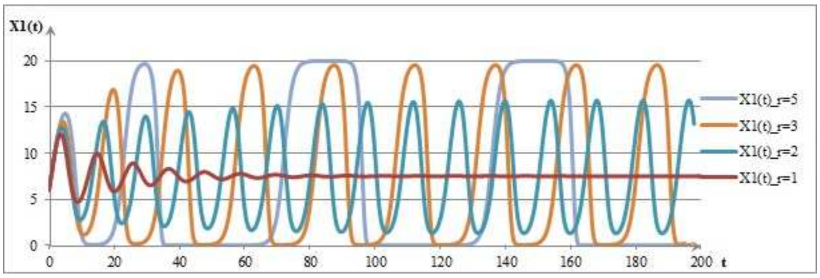

Figure 1: Plot of function $x_1(t)$ depending on the value of delay r

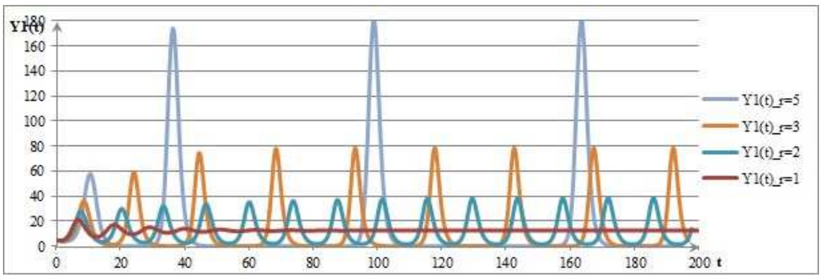

Figure 2: Plot of function $y_1(t)$ depending on the value of delay r

Numerical simulations of the control models are represented with different parameter values. Figures 3 - 4 demonstrate the influence of the delay parameter on the optimal control $\overline{u}_1(t), \overline{v}_1(t)$ and the evolution of the functions $\overline{x}_1(t), \overline{y}_1(t)$ on time interval $t \in [0, T]$.

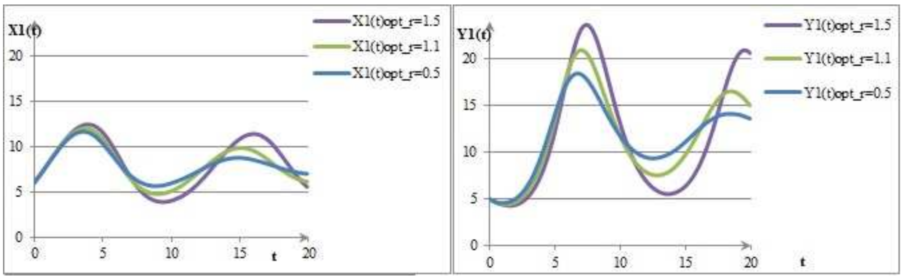

Figure 3: Plot of functions $\overline{x}_1(t)$ and $\overline{y}_1(t)$ depending on the value of delay r

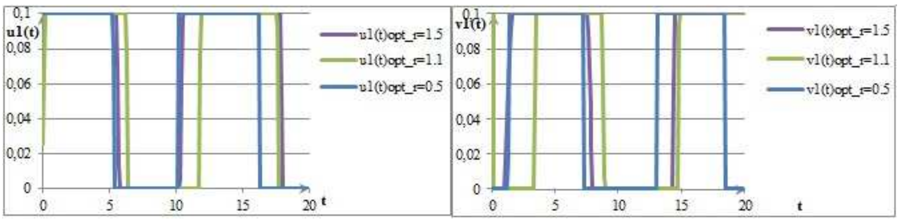

Figure 4: Plot of function $\overline{u}_1(t)$ and $\overline{v}_1(t)$ depending on the value of delay r

The numerical results were received by fast automatic differentiation method, described in [4], [5], and modified genetic algorithm. The classical genetic algorithm was modified because of the large-scale task. Multipoint mutation and multipoint crossover were applied for generating the optimal solution. Software was developed for solving optimal control problems described by the systems of integro-differential equations and differential equations with delay using different techniques. The results of calculations correspond to the theoretical conclusions of maximum principle. Nevertheless, the questions concerned with existence of optimal singular subarcs are open and will be studied in the next papers.

References

[1] Beretta E., Capassa V., Rinaldi F. *Global stability results for a generalized Lotka-Volterra system with distirbed delays.* Journal of Math. Biology, 26, 1988.

[2] Andreeva E. A., Semykina N.A. *Optimal control of the spread of an infection disease with allowance for an incubation period.* Computational Mathematics and mathematical physics, vol 45, 7, 2005, pp. 1133 – 1139.

[3] Andreeva E. A., Pustarnakova Yu. A. *A numerical method for training artificial neural networks with delay.* Computational Mathematics and mathematical physics, vol 42, 9, 2002, pp. 1383 – 1391.

[4] Evtushenko Y. G., Zubov V. I. *Application of the FAD-Methodology for Optimal Control of Melting and Solidification.* Joint EUROPT-OMS Meeting 2007: 2nd Conference on Optimization Methods & Software and 6th EUROPT Workshop on Advances in Continuous Optimization. July 4-7, 2007, Prague, Czech Republic, p. 163.

[5] Evtushenko Y. G., Zasuhina E. S., Zubov V. I. *FAD Method to Compute Second Order Derivatives.* Automatic Differentiation of Algorithms. From Simulation to Optimization, New York: Inc. Springer-Verlag, 2002, pp. 327-333.