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Guided Modes of a Planar Gradient Waveguide

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Abstract. The mathematical model of light propagation in a planar gradient optical waveguide consists of the Maxwell's equations supplemented by the matter equations and boundary conditions. In the coordinates adapted to the waveguide geometry, the Maxwell's equations are separated into two independent sets for the TE and TM polarizations. Each of the systems can be transformed to a second order ordinary differential equation. The boundary conditions for Maxwell's equations are reduced to two pairs of boundary conditions for the obtained equations. Thus, the problem of describing a complete set of modes in a regular planar waveguide is formulated in terms of an eigenvalue problem. For each polarization there are three types of waveguide modes: guided modes, substrate radiation modes, and cover radiation modes. In this work we implement the numerical-analytical calculation of all types of waveguide modes.

Keywords: waveguide propagation of electromagnetic radiation, equations of waveguide modes of regular waveguide, complete set of modes of a planar gradient waveguide

MSC numbers: 65Fxx, 65Hxx, 65L10, 65L15, 78A40, 78Mxx

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1. Introduction

Many planar waveguides used in integrated optics, are asymmetric and have a gradient distribution of refractive index of the waveguide layer as a result of fabrication process. There are exact and approximate analytical methods for simulation of guided modes in planar waveguides with selected elementary profiles of transverse distribution of refractive index in the guiding layer. In the case of an arbitrary piecewise continuous profile the approximate calculation of the electromagnetic field of guided modes is possible only by using numerical methods, implemented on a computer. On the other hand, the numerical finite-difference (or other equivalent) method of solving ODE with arbitrary piecewise continuous coefficients can be implemented only in a finite domain of independent variables. This requirement can be accomplished because in the substrate and in the coating layer with constant refractive indices n_s and n_c, the solutions for guided modes are decaying exponentially. This fact, in turn, enables us to reduce the original Dirichlet problem on the axis to the problem on a segment $[x_1, x_2]$ of the cross-section of the waveguide layer with boundary conditions of the third kind, depending on the spectral parameter [1].

The well proven Galerkin method of numerical solution of the boundary value problem for the second-order ODE (or any other method, generalizing the Galerkin method) uses complete sets of functions. These sets do not satisfy the boundary conditions of the third kind, which depend on the spectral parameter and appear in the formulation of the original problem on the segment. In this case, the classic monographs [2, 3, 4, 5] on numerical methods for solving mathematical physics problems recommend to reduce the problem to finding a zero contribution, satisfying inconvenient or inhomogeneous boundary conditions. The reduced problem with homogeneous boundary conditions of the first kind is recommended to be solved using a wide range of coordinate functions.

In implementing this approach, there are two possibilities: a) $n(x_1) = n(x_2)$, and b) $n(x_1) \neq n(x_2)$. In the case a) as a zero contribution to the solution, one can consider the solution of an idealized thin film waveguide with (piecewise) continuous refractive index distribution in the cross-section of the waveguide layer. This problem is studied in many publications [6] – [13], [18] – [22]. In Ref. [26] there is a detailed description of the complete system of thin-film waveguide modes.

In the case b) as a zero contribution one can consider the solution for the idealized waveguide with linear refractive index distribution $n(x) = ax + b$, where $a = \frac{n(x_2)-n(x_1)}{x_2-x_1}$ and $b = n(x_1)x_2 - n(x_2)x_1$.

In this paper, we will present detailed study of the complete system of modes of a planar gradient waveguide with linear refractive index distribution in the cross-section of the waveguide layer.

To describe the process of propagation of electromagnetic radiation in the integrated optical waveguides using the method of coupled waves [6, 7], the method of comparison waveguides [8, 9] and the incomplete Galerkin method [10, 11] it is necessary to know the complete system of waveguide modes of regular planar comparison reference waveguide [12, 13], and to be able to use them. In this paper we consider a particular, but the most common case of a multilayer waveguide.

2. Formulation of the physical model

In a regular planar gradient optical waveguide, the waveguide modes of three types can propagate: the guided modes, the substrate radiation modes, and the cover radiation modes. A regular waveguide consists of a dielectric waveguide layer (one or more) with the piecewise-linear transverse distribution of refractive index $n_f(x)$ (or $n_{f1}(x)$, ..., $n_{fN}(x)$), surrounded by dielectric media with lower refractive indices: n_s for the substrate layer, and n_c for the cover layer. The Cartesian coordinate system is chosen according to the waveguide structure. The thickness d of the waveguide layer is comparable to the wavelength of monochromatic electromagnetic radiation, the thicknesses of the substrate and the cover layers are significantly greater, and in the model are considered to be infinite.

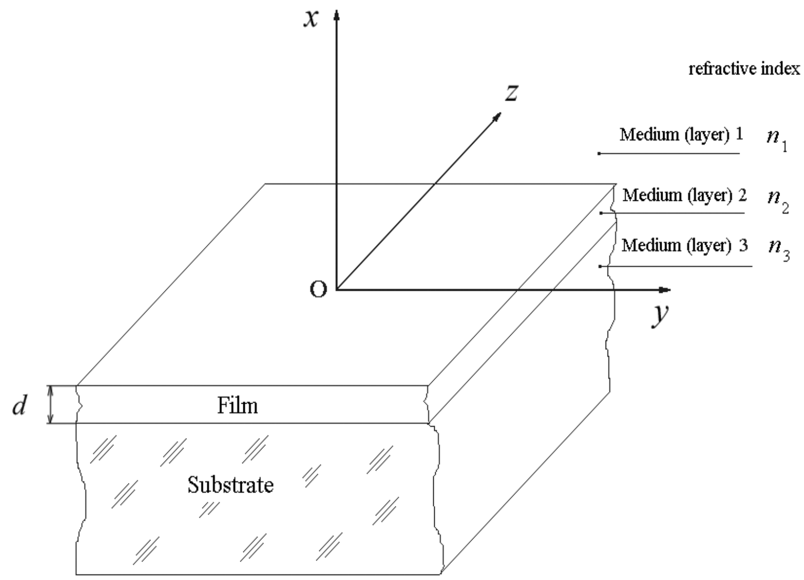

Figure 1: Waveguide formed by media 1−3. The notations are as follows: 1 is a framing medium or cover layer (air) with the refractive index n_c; 2 is a waveguide layer (film) with the refractive index n_f; 3 is a substrate with refractive index n_s; d is the thickness of the waveguide layer. The film and the substrate are homogeneous in the y and z directions, the substrate is usually much thicker than the film.

The Maxwell's equations, supplemented by material equations and boundary conditions are considered as a model of waveguide propagation of light. With the use of the geometry and coordinates of Fig. 1, the Maxwell's equations are reduced to linearly independent subsystems of equations for TE and TM polarizations. The variables are separated, and dependence of electromagnetic field of modes on all independent variables is written in the form:

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} (x, y, z, t) = \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} (x) \exp\left\{i\omega t - ik_0\beta z\right\}. \tag{1}$$

The components TE and TM modes, as functions of x, satisfy the equations:

$$\frac{d^2 E_y}{dx^2} + k_0^2 \left(\varepsilon\mu - \beta^2\right) E_y(x) = 0, \quad H_z = \frac{1}{ik_0\mu}\frac{dE_y}{dx}, \quad H_x = -\frac{\beta}{\mu}E_y,, \tag{2}$$

$$\varepsilon\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{dH_y}{dx}\right) + k_0^2\left(\varepsilon\mu - \beta^2\right) H_y(x) = 0, \quad E_z = -\frac{1}{ik_0\varepsilon}\frac{\partial H_y}{\partial x}, \quad E_x = \frac{\beta}{\varepsilon}H_y,, \tag{3}$$

where ω is the angular frequency, $k_0 = \omega/c$, c is the speed of light in vacuum, and β is the coefficient of phase retardation of the waveguide mode.

Both systems for both mode sets can be written in the "usual" form in the dimensionless variables $\tilde{x} = k_0 x = 2\pi\left(x/\lambda_0\right)$, below written without a tilde:

$$-p(x)\frac{d}{dx}\left(\frac{1}{p(x)}\frac{d\psi}{dx}(k, x)\right) + V(x)\psi(k, x) = k^2\psi(k, x). \tag{4}$$

Here $p(x) = \mu$ for $\psi^{TE}(x) = E_y(k_0 x)$ and $p(x) = \varepsilon$ for $\psi^{TM}(x) = H_y(k_0 x)$, $V(x) = -n^2(k_0 x) = -\varepsilon(k_0 x)\mu$ is a piecewise continuous (continuous in layers) function, $k^2 = -\beta^2$ is the spectral parameter.

The boundary conditions are satisfied for the function $\psi(x)$ and for its "derivative" $\phi^{TE}(x) = \frac{1}{\mu}\frac{d\psi^{TE}}{dx}(x)$ and $\phi^{TM}(x) = \frac{1}{\varepsilon(x)}\frac{d\psi^{TM}}{dx}(x)$:

$$\left.\psi\right|_1 = \left.\psi\right|_2, \quad \left.\phi\right|_1 = \left.\phi\right|_2. \tag{5}$$

The task of finding the waveguide modes consists [14] in finding the eigenvalues k and eigenfunctions $\psi(k, x)$ of the problem (4)–(5) on the axis with the potential $V(x)$, satisfying the asymptotic conditions:

$$V(x) \underset{x\to-\infty}{\longrightarrow} V_s, \; V(x) \underset{x\to\infty}{\longrightarrow} V_c. \tag{6}$$

The problem (4)–(6) in the notation of (2)–(3) in the case of square-integrable functions, i.e. in the case of a discrete spectrum $k_j = i\kappa_j$ for piecewise constant potential $V(x)$ shown in Fig. 2 (for multilayer waveguide), is a subject of many studies, both theoretical and computational (see, e.g., the fundamental works on integrated optics [15]–[17] and books on integrated optics [18]–[22], devoted to the description of guided waveguide modes). Numerical methods of constructing the eigenfunctions $\psi_j(x)$ via the expansion in fundamental sets of solutions of the equation (4) in each layer and linking them at the interfaces of the layers according to the relations (5) are implemented in the basic papers on integrated optics [15]–[17] and in recent papers [23]–[25].

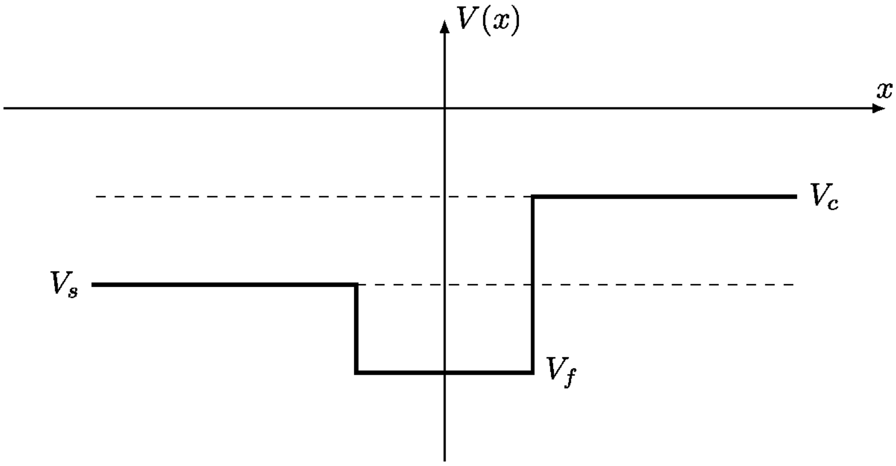

Figure 2: Schematic diagram of the potential $V(x)$ in the case of multilayer waveguide.

3. Statement of the problem

The problem of describing the full set of waveguide modes of a regular gradient planar optical waveguide is formulated as an eigenvalue problem (for discrete and continuous spectra) and eigenfunction problem (for classical and generalized functions) of essentially self-adjoint ordinary differential operator of the second order [14, 26]:

$$-p(x)\frac{d}{dx}\left(\frac{1}{p(x)}\frac{d\psi}{dx}(k,x)\right)+V(x)\psi(k,x)=k^2\psi(k,x). \tag{7}$$

Here $p(x)=\mu$ for $\psi^{TE}(x)=E_y(k_0x)$ and $p(x)=\varepsilon$ for $\psi^{TM}(x)=H_y(k_0x)$, $V(x)=-n^2(k_0x)=-\varepsilon(k_0x)\mu$ is piecewise-linear (linear in layers) function, $k^2=-\beta^2$ is the spectral parameter.

With the auxiliary functions

$$\phi_{TE}(x)=\frac{1}{p(x)}\frac{d\psi_{TE}}{dx}(x), \quad \phi_{TM}(x)=\frac{1}{p(x)}\frac{d\psi_{TM}}{dx}(x) \tag{8}$$

we can write down the reduced boundary conditions at the points of discontinuity of the potential, and therefore of the second derivative of the solution:

$$\psi|_{x_1-0}=\psi|_{x_1+0}, \quad \psi|_{x_2-0}=\psi|_{x_2+0}, \tag{9}$$

$$\phi|_{x_1-0}=\phi|_{x_1+0}, \quad \phi|_{x_2-0}=\phi|_{x_2+0}. \tag{10}$$

Besides, the asymptotic conditions are satisfied

$$|\psi(x)|_{x\to\pm\infty}\le C^{\pm}. \tag{11}$$

The spectrum of operator (7)–(11) consists of [12, 13]:

a finite number of discrete eigenvalues $k_j = i\kappa_j : \ k_j^2 \in (\min V(x), \ \min(V_s, \ V_c))$ and the corresponding classical eigenfunctions (of guided waveguide modes);

a single continuous spectrum $k_s : \ k_s^2 \in (V_s, \ \infty)$ and the corresponding generalized eigenfunctions (substrate radiation modes);

a single continuous spectrum $k_c : \ k_c^2 \in (V_c, \ \infty)$ and the corresponding generalized eigenfunctions (cover radiation modes).

For a constructive description of the problem solutions, i.e. eigenfunctions of three types, we shall restrict our consideration to piecewise-linear potential:

$$V(x) = \begin{cases} V_s, & \text{when } x < x_1 \\ ax + b, & \text{when } x_1 < x < x_2 \\ V_c, & \text{when } x > x_2 \end{cases}, \text{where } a = \frac{V_2 - V_1}{x_2 - x_1}, b = \frac{V_1 x_2 - V_2 x_1}{x_2 - x_1}.$$

The function $V(x)$ has the view shown in Fig. 3.

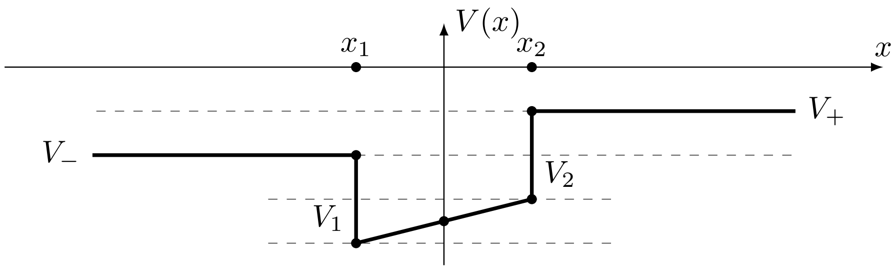

Figure 3: Schematic diagram of the potential $V(x)$ in the case of gradient waveguide.

4 The solution for the eigenvalues (of the discrete spectrum) and eigenfunctions (classical)

The method of solution is the expansion on the sub-intervals of the general solution in terms of the fundamental system of solutions.

In the region $(-\infty, x_1)$ the general solutions of the equation (7) with constant coefficient V_s and satisfying the asymptotic condition $\psi(x) \underset{x\to-\infty}{\longrightarrow} 0$ are of the form (for TE and TM modes, respectively):

$$\psi_s^{TE}(x) = C_s \exp\{\gamma_s (x - x_1)\}, \phi_s^{TE}(x) = \frac{\gamma_s}{\mu_s} C_s \exp\{\gamma_s (x - x_1)\}, \quad (12)$$

$$\gamma_s = \sqrt{V_s - k^2} > 0, \quad (13)$$

$$\psi_s^{TM}(x) = D_s \exp\{\gamma_s (x - x_1)\}, \phi_s^{TM}(x) = \frac{\gamma_s}{\varepsilon_s} D_s \exp\{\gamma_s (x - x_1)\}, \quad (14)$$

$$\gamma_s = \sqrt{V_s - k^2} > 0. \quad (15)$$

In the region (x_2, ∞) the general solutions of the equation (7), satisfying the asymptotic condition $\psi(x) \underset{x\to\infty}{\longrightarrow} 0$, have the form:

$$\psi_c^{TE}(x) = C_c \exp\{\gamma_c (x - x_2)\}\,,\ \phi_c^{TE}(x) = -\frac{\gamma_c}{\mu_c} C_c \exp\{\gamma_c (x - x_2)\}\,, \tag{16}$$

$$\gamma_c = \sqrt{V_c - k^2} > 0\,, \tag{17}$$

$$\psi_c^{TM}(x) = D_c \exp\{\gamma_c (x - x_2)\}\,,\ \phi_c^{TM}(x) = -\frac{\gamma_c}{\varepsilon_c} D_c \exp\{\gamma_c (x - x_2)\}\,, \tag{18}$$

$$\gamma_c = \sqrt{V_c - k^2} > 0\,. \tag{19}$$

In the waveguide layer (with a linear potential in the subdomain) the fundamental system of solutions consists of the functions $Ai(x)$ and $Bi(x)$. In the region (x_1, x_2) the general solutions of the equation (7) have the form (for TE and TM modes, respectively):

$$\Psi_f^{TE}(k,x) = C_1 \mathrm{Ai}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) + C_2 \mathrm{Bi}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right), \tag{20}$$

$$\Phi_f^{TE}(k,x) = -C_1 \frac{(-a)^{1/3}}{\mu}\frac{\mathrm{dAi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) - \\ - C_2 \frac{(-a)^{1/3}}{\mu}\frac{\mathrm{dBi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right), \tag{21}$$

$$\Psi_f^{TM}(k,x) = D_1 \mathrm{Ai}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) + D_2 \mathrm{Bi}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right), \tag{22}$$

$$\Phi_f^{TM}(k,x) = -D_1 \frac{(-a)^{1/3}}{\varepsilon}\frac{\mathrm{dAi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) - \\ - D_2 \frac{(-a)^{1/3}}{\varepsilon}\frac{\mathrm{dBi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right). \tag{23}$$

These common solutions in the subdomains form a single particular solution of the problem (7)–(11), therefore, the equalities must be satisfied:

$$\Psi_s(k, x_1) = \Psi_f(k, x_1)\,, \quad \Phi_s(k, x_1) = \Phi_f(k, x_1)\,,$$

$$\Psi_f(k, x_2) = \Psi_c(k, x_2)\,, \quad \Phi_f(k, x_2) = \Phi_c(k, x_2)\,.$$

Thus we obtain a homogeneous system of linear algebraic equations for the indefinite coefficients of the expansion of common solutions in terms of the fundamental systems of solutions, which for the TE modes has the form:

$$C_s = C_1 Ai\left(\frac{-ad+b}{(-a)^{2/3}}\right) + C_2 Bi\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$\gamma_s \frac{1}{\mu} C_s = -C_1 \frac{1}{\mu} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right) - C_2 \frac{1}{\mu} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$C_1 Ai \left(\frac{b}{(-a)^{2/3}} \right) + C_2 Bi \left(\frac{b}{(-a)^{2/3}} \right) = C_c,$$

$$-C_1 \frac{1}{\mu} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) - C_2 \frac{1}{\mu} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) = -\gamma_c \frac{1}{\mu} C_c.$$

The resulting homogeneous system of linear algebraic equations

$$\hat{M}_{TE}(k) \vec{C}(k) = \vec{0} \tag{24}$$

has a non-trivial solution provided that

$$\det \hat{M}_{TE}(k) = 0. \tag{25}$$

The homogeneous system of linear algebraic equations for the unknown coefficients of the expansion of common solutions in terms of the fundamental system of solutions for the TM modes has the form:

$$D_s = D_1 Ai \left(\frac{-ad+b}{(-a)^{2/3}} \right) + D_2 Bi \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$\gamma_s \frac{1}{\varepsilon} D_s = -D_1 \frac{1}{\varepsilon} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right) - D_2 \frac{1}{\varepsilon} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$D_1 Ai \left(\frac{b}{(-a)^{2/3}} \right) + D_2 Bi \left(\frac{b}{(-a)^{2/3}} \right) = D_c,$$

$$-D_1 \frac{1}{\varepsilon} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) - D_2 \frac{1}{\varepsilon} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) = -\gamma_c \frac{1}{\varepsilon} D_c.$$

The resulting homogeneous system of linear algebraic equations

$$\hat{M}_{TM}(k) \vec{D}(k) = \vec{0} \tag{26}$$

has a non-trivial solution provided that

$$\det \hat{M}_{TM}(k) = 0. \tag{27}$$

The solutions k_j^{TE} of the nonlinear transcendental algebraic equation (25)are substituted into the SLAE (24) and then this system is solved with respect to $\vec{C}_j = \vec{C}(k_j)$.

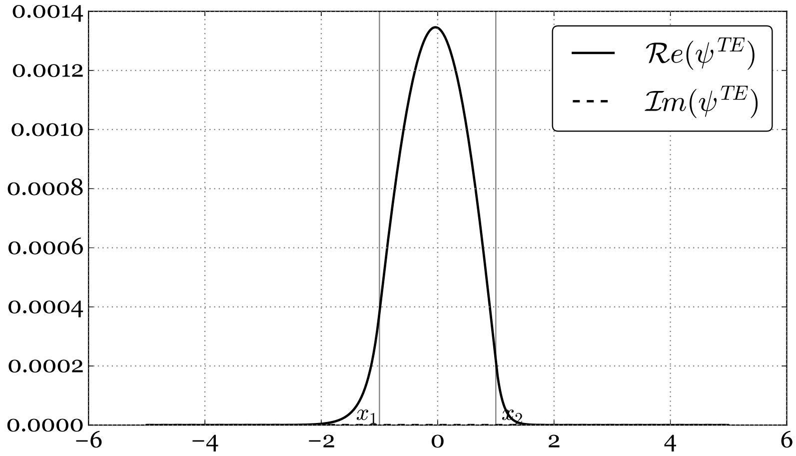

Figure 4: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the guided modes TE_0, $n_c = 1.0$, $n_f = 2.15$, $n_s = 1.515$, $\beta = 1.6752$.

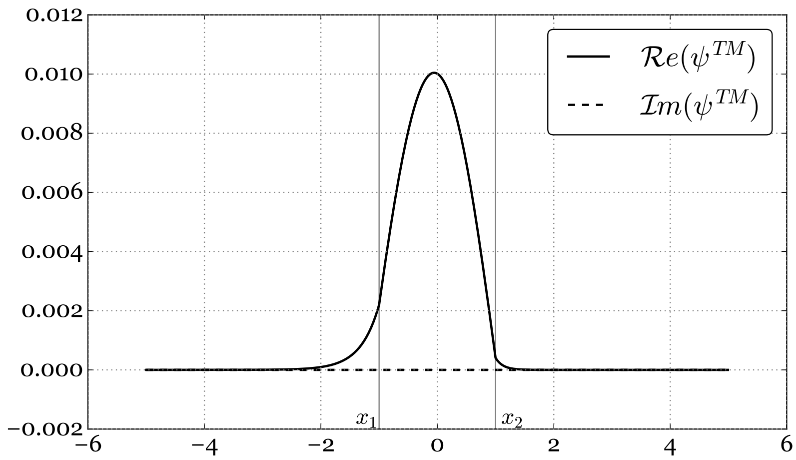

Figure 5: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the guided modes TM_0, $n_c = 1.0$, $n_f = 2.15$, $n_s = 1.515$, $\beta = 1.5955$.

The obtained coefficients are substituted in the expressions (12), (16), (20), (21) for the fields. The results of calculations are presented in Fig. 4.

The solutions k_j^{TM} of the nonlinear transcendental algebraic equation (27) are substituted in the SLAE (26) and then this system is solved with respect to $\vec{D}_j = \vec{D}\left(k_j\right)$. The obtained coefficients are substituted in the expressions (14), (18), (22), (23) for the fields. The results of calculations are presented in Fig. 4.

The results and the method for calculating the eigenvalues k_j^{TE} and eigenvectors $\left(A_s, A_f^+, A_f^-, A_c\right)^T$, as well as k_j^{TM} and $\left(B_s, B_f^+, B_f^-, B_c\right)^T$ for thin-film planar optical waveguides are described in publications [15]–[17] and monographs [18]–[22]. These works also report the results of many numerical experiments. A brief presentation of the method can be found in [26, 27]. There one can find also numerical implementations on a computer of dispersion relations for TE modes and corresponding distributions with respect to Ox axis of electric and magnetic fields of nonzero components for TE modes.

5. Calculation of cover radiation modes

Similarly to what was done in [14, 26] for piecewise-constant potentials, let's move from the solutions of the problem (7) - (11) satisfying the asymptotic Jost conditions, to the solutions satisfying the "scattering problem" conditions. A one-to-one correspondence between them is set in [14, 26] for the potentials of a more general kind.

In particular, the asymptotic behavior of the cover radiation modes $\psi_c\left(k, x\right)$ correspond to the problem of scattering of a plane Jost wave incident on the potential $V\left(x\right)$ from the right, which is partially reflected back to the right with reflection coefficient $R_-\left(k\right)$, and partially transmitted (through the potential $V\left(x\right)$) to the left with the transmittance coefficient $T_-\left(k\right)$, taking the form of a plane Jost wave propagating from right to left in the region $x > -\infty$. All solutions $\psi_c\left(k, x\right)$ satisfy these asymptotic conditions when $k^2 \in \left(V_c, \infty\right)$. The sought solution, as in the case of guided modes, is constructed by joining at the boundaries of the general solutions of the equation (8) in the regions of the argument $\left(-\infty, x_1\right)$, $\left(x_1, x_2\right)$, and $\left(x_2, \infty\right)$.

Hence, in the region $\left(-\infty, x_1\right)$ the general solutions of Eq. (7) with the constant coefficient V_s and the auxiliary functions have the form (for TE and TM modes, respectively):

$$\psi_{c,s}^{TE}\left(k, x\right) = T_c^{TE}\left(k\right) \exp\left\{-ip_s\left(x - x_1\right)\right\},$$
$$\phi_{c,s}^{TE}\left(k, x\right) = -\frac{ip_s}{\mu_s} T_c^{TE}\left(k\right) \exp\left\{-ip_s\left(x - x_1\right)\right\},$$

$$\psi_{c,s}^{TM}\left(k, x\right) = T_c^{TM}\left(k\right) \exp\left\{-ip_s\left(x - x_1\right)\right\},$$
$$\phi_{c,s}^{TM}\left(k, x\right) = -\frac{ip_s}{\varepsilon_s} T_c^{TM}\left(k\right) \exp\left\{-ip_s\left(x - x_1\right)\right\}.$$

In the region (x_1, x_2) the general solutions of the equation (7) and the auxiliary functions have the form (for TE and TM modes, respectively):

$$\Psi^{TE}_{c,f}(k,x) = C^1_c \frac{1}{\mu} \mathrm{Ai}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) + C^2_c \frac{1}{\mu} \mathrm{Bi}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right),$$

$$\Phi^{TE}_{c,f}(k,x) = -C^1_c \frac{1}{\mu}(-a)^{1/3} \frac{\mathrm{dAi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) - \\ - C^2_c \frac{1}{\mu}(-a)^{1/3} \frac{\mathrm{dBi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right), \quad (28)$$

$$\Psi^{TM}_{c,f}(k,x) = D^1_c \mathrm{Ai}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) + D^2_c \mathrm{Bi}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right),$$

$$\Phi^{TM}_{c,f}(k,x) = -D^1_c (-a)^{1/3} \frac{\mathrm{dAi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) - \\ - D^2_c (-a)^{1/3} \frac{\mathrm{dBi}}{\mathrm{d}x}\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right). \quad (29)$$

Thus, the solutions (for TE modes) are given by the sets of amplitude coefficients $(T^{TE}_c, C^1_c, C^2_c, R^{TE}_c)^T$ satisfying the system of linear algebraic equations:

$$T^{TE}_c(k) = C^1_c Ai\left(\frac{-ad+b}{(-a)^{2/3}}\right) + C^2_c Bi\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$-\frac{ip_s}{\mu_s} T^{TE}_{-}(k) = -C^1_f \frac{1}{\mu}(-a)^{1/3} \frac{dAi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right) - C^2_f \frac{1}{\mu}(-a)^{1/3} \frac{dBi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$C^1_c Ai\left(\frac{b}{(-a)^{2/3}}\right) + C^2_c Bi\left(\frac{b}{(-a)^{2/3}}\right) = 1 + R^{TE}_c(k),$$

$$-C^1_c \frac{1}{\mu}(-a)^{1/3} \frac{dAi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) - C^2_c \frac{1}{\mu}(-a)^{1/3} \frac{dBi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) = \\ = -\frac{ip_c}{\mu_c}\left[1 - R^{TE}_c(k)\right].$$

The solutions for TM modes are given by the sets of amplitude coefficients $\left(T^{TM}_c, D^1_c, D^2_c, R^{TM}_c\right)^T$ satisfying the system of linear algebraic equations:

$$T^{TM}_c(k) = D^1_c Ai\left(\frac{-ad+b}{(-a)^{2/3}}\right) + D^2_c Bi\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$-\frac{ip_s}{\varepsilon_s}T_c^{TM}(k) = -D_f^1\frac{1}{\varepsilon}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right) - D_f^2\frac{1}{\varepsilon}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$D_c^1 Ai\left(\frac{b}{(-a)^{2/3}}\right) + D_c^2 Bi\left(\frac{b}{(-a)^{2/3}}\right) = 1 + R_c^{TM}(k),$$

$$-D_c^1\frac{1}{\varepsilon}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) - D_c^2\frac{1}{\varepsilon}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) = $$

$$= -\frac{ip_c}{\varepsilon_c}\left[1 - R_c^{TM}(k)\right].$$

In both cases, we arrive at the inhomogeneous SLAE of the form:

$$\hat{M}^{TE}(k)\left(T_c^{TE}, C_c^1, C_c^2, R_c^{TE}\right)^T = \left(0, 0, 1, -\frac{ip_c}{\mu_c}\right)^T,$$

$$\hat{M}^{TM}(k)\left(T_c^{TM}, D_c^1, D_c^2, R_c^{TM}\right)^T = \left(0, 0, 1, -\frac{ip_c}{\varepsilon_c}\right)^T,$$

so that the solution exists for any $k^2 \in (V_c, \infty)$ and is unique up to a complex factor (see Fig. 6 for the case of TE-modes). The plots of the solutions for TM-modes qualitatively repeat the plots shown in Fig. 7.

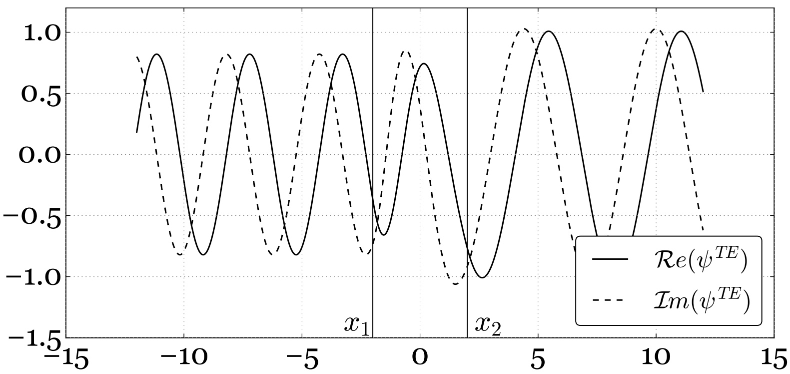

Figure 6: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the cover radiation modes, $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = 0.250$

Cover radiation modes are described in [12, 13, 28, 29], as generalized eigensolutions of Eq. (7) with the boundary conditions (9), (10) for the values of spectral parameter $k \in (V_c, \infty)$. They are, as in the case of guided modes, built by linking at the boundaries of the general solutions of Eq. (7) in the argument regions $(-\infty, x_1)$, (x_1, x_2), and (x_2, ∞). The solution of the problem of scattering on the potential $V(x)$ with similar asymptotic behavior is described in [31, 32].

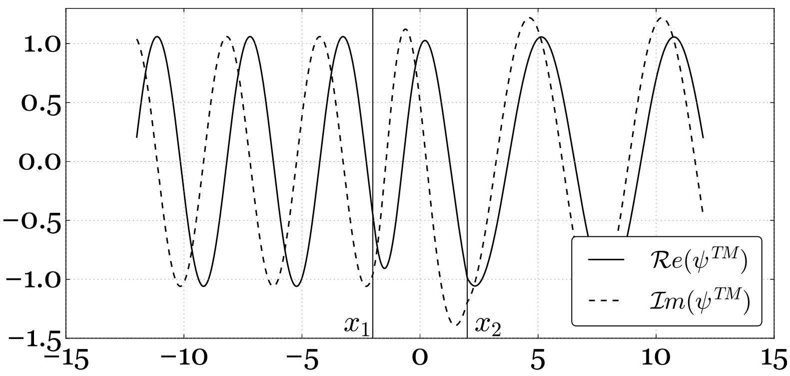

Figure 7: The curves for the field strength (along the vertical axis) corresponding to the first spectral values for the cover radiation modes, $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = 0.250$.

6. Calculation of substrate radiation modes

The asymptotic behavior of the substrate radiation modes $\psi_s(k,x)$ correspond to the scattering of a plane Jost wave incident from the left, on the potential $V(x)$. The wave is partially reflected to the left with the reflection coefficient $R_s(k)$. At the same time, the Jost wave coming from the left, passing through the potential $V(x)$, propagates to the right as the plane Jost wave with the transmittance coefficient $T_s(k)$ when $k^2 \in (V_c, \infty)$, and as an evanescent wave decaying to the right with a weighting factor $C_c(k)(D_c(k))$ when $k^2 \in (V_s, V_c)$.

The solutions have different form for different values of the spectral parameter k belonging to the spectral subregions $k^2 \in (V_s, V_c)$ and $k^2 \in (V_c, \infty)$. But for both regions the solution, as in the case of guided modes, is constructed by joining at the boundaries of the general solutions of Eq. (7) in the regions of the argument $(-\infty, x_1)$, (x_1, x_2), and (x_2, ∞).

In the region $(-\infty, x_1)$ the general solutions of Eq. (7) with the spectral parameter $k^2 \in (V_s, V_c)$ have the form:

$$\psi_{s,s}^{TE}(k,x) = \exp\{ip_s(k)(x-x_1)\} + R_s^{TE}(k)\exp\{-ip_s(k)(x-x_1)\},$$

$$\phi_{s,s}^{TE}(k,x) = \frac{ip_s}{\mu_s}\left[\exp\{ip_s(k)(x-x_1)\} - R_s^{TE}(k)\exp\{-ip_s(k)(x-x_1)\}\right],$$

$$\psi_{s,s}^{TM}(k,x) = \exp\{ip_s(k)(x-x_1)\} + R_s^{TM}(k)\exp\{-ip_s(k)(x-x_1)\},$$

$$\phi_{s,s}^{TM}(k,x) = \frac{ip_s}{\varepsilon_s}\left[\exp\{ip_s(k)(x-x_1)\} - R_s^{TM}(k)\exp\{-ip_s(k)(x-x_1)\}\right].$$

In the region (x_1, b) the general solutions of Eq. (7) with the spectral parameter $k^2 \in (V_s, V_c)$ have the form:

$$\Psi_{s,f}^{TE}(k,x) = C_s^1 Ai\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right) + C_s^2 Bi\left(\frac{a(x-x_2)+b}{(-a)^{2/3}}\right),$$

$$\Phi^{TE}_{s,f}(k,x) = -C^1_s \frac{(-a)^{1/3}}{\mu} \frac{\mathrm{dAi}}{\mathrm{d}x} \left(\frac{a(x-x_2)+x_2}{(-a)^{2/3}} \right) - \\ - C^2_s \frac{(-a)^{1/3}}{\mu} \frac{\mathrm{dBi}}{\mathrm{d}x} \left(\frac{a(x-x_2)+b}{(-a)^{2/3}} \right), \quad (30)$$

$$\Psi^{TM}_{s,f}(k,x) = D^1_s Ai \left(\frac{a(x-x_2)+b}{(-a)^{2/3}} \right) + D^2_s Bi \left(\frac{a(x-x_2)+b}{(-a)^{2/3}} \right),$$

$$\Phi^{TM}_{s,f}(k,x) = -D^1_s \frac{(-a)^{1/3}}{\varepsilon_f} \frac{\mathrm{dAi}}{\mathrm{d}x} \left(\frac{a(x-x_2)+x_2}{(-a)^{2/3}} \right) - \\ - C^2_s \frac{(-a)^{1/3}}{\varepsilon_f} \frac{\mathrm{dBi}}{\mathrm{d}x} \left(\frac{a(x-x_2)+b}{(-a)^{2/3}} \right). \quad (31)$$

In the region (x_2, ∞) the general solutions of Eq. (7) with the spectral parameter $k^2 \in (V_s, V_c)$ have the form (by virtue of the asymptotic decay at infinity):

$$\psi^{TE}_{s,c}(k,x) = C^{TE}_s \exp\{-\gamma_c (x-x_2)\}, \quad \phi^{TE}_{s,c}(k,x) = -\frac{\gamma_c}{\mu_c} C^{TE}_s \exp\{-\gamma_c (x-x_2)\},$$

$$\psi^{TM}_{s,c}(k,x) = D^{TM}_s \exp\{-\gamma_c (x-x_2)\}, \quad \phi^{TM}_{s,c}(k,x) = -\frac{\gamma_c}{\varepsilon_c} D^{TM}_s \exp\{-\gamma_c (x-x_2)\}.$$

Thus, the solutions (for TE and TM modes, respectively) are given by the sets of amplitude coefficients $\left(R^{TE}_s, C^1_s, C^2_s, C^{TE}_s\right)^T$ satisfying the system of linear algebraic equations:

$$1 + R^{TE}_s(k) = C^1_s Ai \left(\frac{-ad+b}{(-a)^{2/3}} \right) + C^2_s Bi \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$\frac{ip_s}{\mu_s}\left[1 - R^{TE}_s(k)\right] = -\frac{C^1_s}{\mu} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right) - \frac{C^2_s}{\mu} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$C^1_s Ai \left(\frac{b}{(-a)^{2/3}} \right) + C^2_s Bi \left(\frac{b}{(-a)^{2/3}} \right) = C^{TE}_s,$$

$$-C^1_s \frac{1}{\mu} (-a)^{1/3} \frac{dAi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) - C^2_s \frac{1}{\mu} (-a)^{1/3} \frac{dBi}{dx} \left(\frac{b}{(-a)^{2/3}} \right) = -\frac{\gamma_c}{\mu_c} C^{TE}_s,$$

and the coefficients $\left(R^{TM}_s, D^1_s, D^2_s, D^{TM}_s\right)^T$ satisfying the system of linear algebraic equations:

$$1 + R^{TM}_s(k) = D^1_s Ai \left(\frac{-ad+b}{(-a)^{2/3}} \right) + D^2_s Bi \left(\frac{-ad+b}{(-a)^{2/3}} \right),$$

$$\frac{ip_s}{\varepsilon_s}\left[1-R_s^{TM}(k)\right] = -\frac{D_s^1}{\varepsilon}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right) - \frac{D_s^2}{\varepsilon}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{-ad+b}{(-a)^{2/3}}\right),$$

$$D_s^1 Ai\left(\frac{b}{(-a)^{2/3}}\right) + D_s^2 Bi\left(\frac{b}{(-a)^{2/3}}\right) = D_s^{TM},$$

$$-D_s^1\frac{1}{\varepsilon_f}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) - D_s^2\frac{1}{\varepsilon_f}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) = -\frac{\gamma_c}{\varepsilon_c}D_s^{TM}.$$

In both cases we arrive at the inhomogeneous SLAE of the form

$$\hat{M}^{TE}(k)\left(R_s^{TE}, C_s^1, C_s^2, C_s^{TE}\right)^T = \left(1, \frac{ip_s}{\mu_s}, 0, 0\right)^T,$$

$$\hat{M}^{TM}(k)\left(R_s^{TM}, D_s^1, D_s^2, D_s^{TM}\right)^T = \left(1, \frac{ip_s}{\varepsilon_s}, 0, 0\right)^T,$$

so that there exists a solution for any $k^2 \in (V_s, V_c)$ and it is unique up to a complex multiplier (Fig. 8, 9).

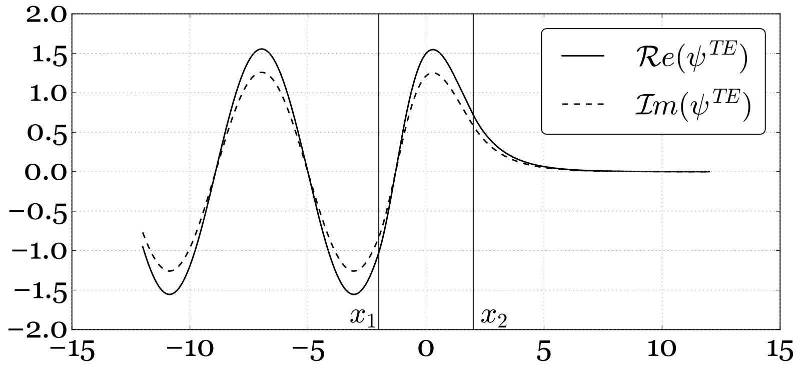

Figure 8: The curves for the field strength (along the vertical axis) for the substrate radiation modes decaying in the cover layer: $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = -1.648$.

For the spectral parameter k from the region $k^2 \in (V_c, \infty)$, in the coordinate regions $(-\infty, x_1)$ and (x_1, x_2) the common solutions have the same form as in the case $k^2 \in (V_s, V_c)$, and in the region (x_2, ∞), they take the form:

$$\psi_{s,c}^{TE}(k,x) = T_s^{TE}(k)\exp\{ip_c(k)(x-x_2)\},$$

$$,\phi_{s,c}^{TE}(k,x) = \frac{ip_c(k)}{\mu_c}T_s^{TE}(k)\exp\{ip_c(k)(x-x_2)\},$$

$$\psi_{s,c}^{TM}(k,x) = T_s^{TM}(k)\exp\{ip_c(k)(x-x_2)\},$$

$$\phi_{s,c}^{TM}(k,x) = \frac{ip_c(k)}{\varepsilon_c}T_s^{TM}(k)\exp\{ip_c(k)(x-x_2)\}.$$

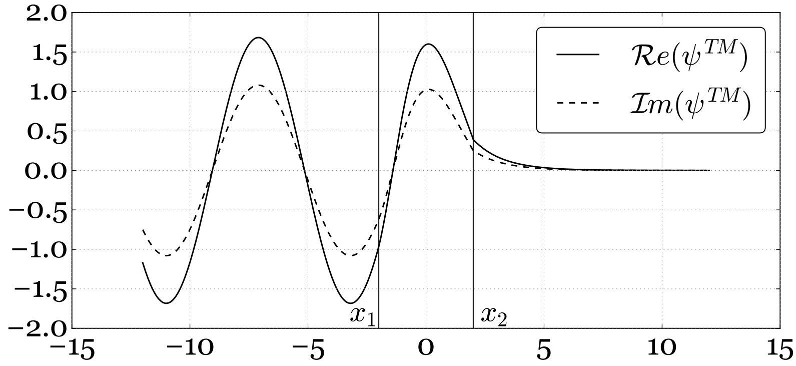

Figure 9: The curves for the field strength (along the vertical axis) for the substrate radiation modes decaying in the cover layer: $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = -1.648$.

Consequently, the second pair of boundary equations at the point $x = x_2$ for TE modes takes the form:

$$C_s^1 Ai\left(\frac{b}{(-a)^{2/3}}\right) + C_s^2 Bi\left(\frac{b}{(-a)^{2/3}}\right) = T_s^{TE}(k),$$

$$-C_s^1 \frac{1}{\mu}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) - C_s^2 \frac{1}{\mu}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) = \frac{ip_c(k)}{\mu_c}T_s^{TE}(k).$$

The resulting SLAE can be rewritten as:

$$\hat{M}^{TE}(k)\left(R_s^{TE}, C_s^1, C_s^2, T_s^{TE}\right)^T = \left(1, \frac{ip_s}{\mu_s}, 0, 0\right)^T,$$

so that there exists a solution for any $k^2 \in (V_c, \infty)$ and it is unique up to a complex multiplier (Fig. 10).

The second pair of boundary equations at the point $x = x_2$ for TM modes take the form:

$$D_s^1 Ai\left(\frac{b}{(-a)^{2/3}}\right) + D_s^2 Bi\left(\frac{b}{(-a)^{2/3}}\right) = \frac{ip_c(k)}{\varepsilon_c}T_s^{TM}(k),$$

$$-\frac{D_s^1}{\varepsilon_f}(-a)^{1/3}\frac{dAi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) - \frac{D_s^2}{\varepsilon_f}(-a)^{1/3}\frac{dBi}{dx}\left(\frac{b}{(-a)^{2/3}}\right) = \frac{ip_c(k)}{\varepsilon_c}T_s^{TM}(k).$$

The resulting SLAE can be rewritten as

$$\hat{M}^{TM}(k)\left(R_s^{TM}, C_s^1, C_s^2, T_s^{TM}\right)^T = \left(1, \frac{ip_s}{\varepsilon_s}, 0, 0\right)^T,$$

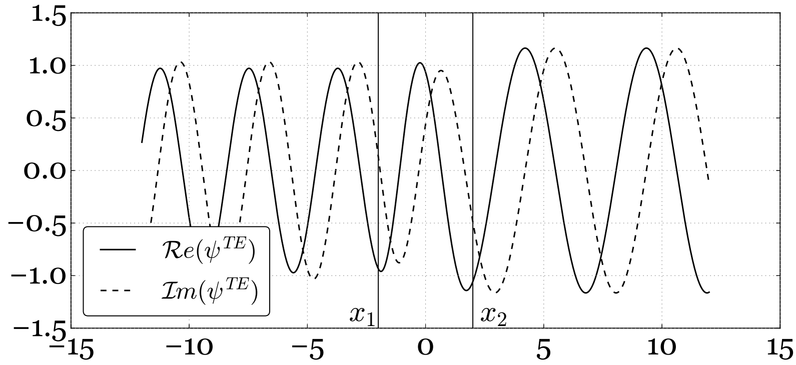

Figure 10: The curves for the field strength (along the vertical axis) for the substrate radiation modes oscillating in the cover layer: $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = 0.250$.

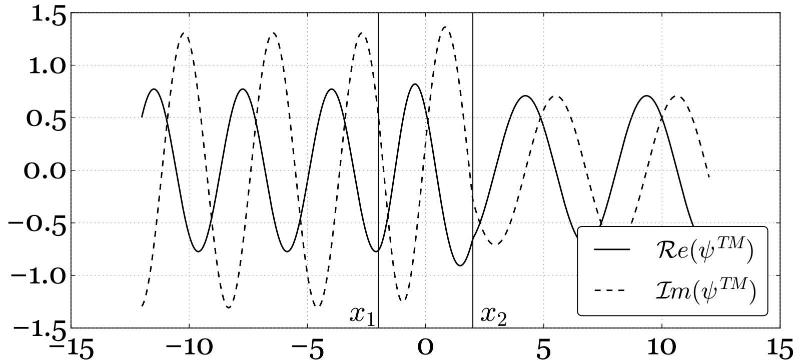

Figure 11: The curves for the field strength (along the vertical axis) for the substrate radiation modes oscillating in the cover layer: $n_c = 1.0$, $n_f = 1.59$, $n_s = 1.515$, $k^2 = 0.250$.

so that there exists a solution for any $k^2 \in (V_c, \infty)$ and it is unique up to a complex multiplier factor. The solutions of this system qualitatively repeat the solutions shown in Fig. 11.

7. Conclusion

The solution of many problems of integrated optics includes spectral analysis and spectral synthesis for a complete system of solutions for second-order differential operator defining waveguide modes of an open waveguide. In the simplest case of a regular waveguide the operator is essentially self-adjoint and has a mixed spectrum: the final single discrete spectrum and two branches of the continuous spectrum [12, 13]. This full system of modes is used to describe the waveguide propagation of electromagnetic radiation using methods of reference waveguides, which can also be used to implement the incomplete Galerkin method in integrated optical waveguides.

This paper presents the numerical implementations of square-integrable eigenfunctions corresponding to the discrete spectrum $k_j = i\kappa_j$ for a piecewise-linear potential $V(x)$ (for the gradient waveguide). The present study also shows the numerical computer implementations of the cover radiation modes and substrate radiation modes. For modeling these modes, the problems of scattering on the potential $V(x)$ of Jost functions equivalent to the original problem in the case of the continuous spectrum were used: the problems of scattering on the left for the substrate radiation modes and the problems of scattering on the right for the cover radiation modes.

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