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Local invariants of smooth foliations

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Abstract. We study local geometry of foliations generated by smooth submersions. The canonical form of the structure equations of a smooth submersion has been obtained. As an example we consider the foliation of two-dimensional surfaces in three-dimensional Euclidean space and describe the invariants of the foliation.

Keywords: smooth foliation, differential invariant, algebra of invariants, structure equations, frame bundle, Elie Cartan's method

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1. Introduction

The geometry of smooth foliations was studied in [1]–[8] by one of the authors of this article. He used the methods described in [9], [10]. In this paper, we show how to investigate the geometry of smooth foliations by the classical Elie Cartan method of external forms and moving frame. The method was enhanced by a number of geometers, especially German Laptev, see, e.g. [11]–[13]. In particular, in [13] the invariant theory of differentiable mappings of a smooth manifold into a manifold of higher dimension was constructed. We develop the theory for smooth submersions. At the beginning we provide the canonical form of the structure equations of a smooth submersion, then we show that G-structures of the first and second orders and some three-valent tensor canonically associated with a submersion. In the second part, we consider in detail the foliation of two-dimensional surfaces in three-dimensional Euclidean space. We find the differential invariants of the foliation and explain their geometric meaning. In particular, it is shown that the algebra of invariants of this foliation is generated by the invariants of the second differential neighbourhood.

2. The canonical structure equations of a smooth submersion

Let M and X be the smooth manifolds of dimensions n and r respectively, $n > r$ and $f : M \to X$ be a smooth map (submersion). Following [11] we write the structure equations of the manifold M in the form

$$\begin{aligned} d\omega^i =&\omega^j \wedge \omega^i_j, \\ d\omega^i_j =&\omega^k_j \wedge \omega^i_k + \omega^k \wedge \omega^i_{jk}, \\ d\omega^i_{jk} =&\omega^m_{jk} \wedge \omega^i_m - \omega^i_{mk} \wedge \omega^m_j - \omega^i_{jm} \wedge \omega^m_k + \omega^m \wedge \omega^i_{jkm}, \quad \dots \end{aligned} \tag{1}$$

Here, ω^i, $i, j, k, m, ... = 1, 2, \dots n$, are the basic differential forms of the manifold M depending on the differentials dx^i, where x^i are the local coordinates on M.

It is known [12], that the forms ω^i and ω^i_j form the basis of the bundle $H^1(M)$ of first-order coframes of M; the forms $\omega^i, \omega^i_j, \omega^i_{jk}$ form the basis of the bundle $H^2(M)$ of the second order coframes of M, etc.

Similarly, we write the structure equations of the manifold X:

$$\begin{aligned} d\vartheta^a =&\vartheta^b \wedge \vartheta^a_b, \\ d\vartheta^a_b =&\vartheta^c_b \wedge \vartheta^a_c + \vartheta^c \wedge \vartheta^a_{bc}, \\ &\dots \end{aligned} \tag{2}$$

Here, ϑ^a, $a, b, c, ... = 1, 2, \dots m$, are the basic differential forms of the manifold X depending on the differentials du^a, where u^a are the local coordinates on X.

In local coordinates, the submersion f can be given by the equations $u = f(x)$. By differentiating this equation and replacing the differentials of variables on the invariant forms ω^i and ϑ^a, we obtain the differential equation of the submersion f in the invariant form

$$\vartheta^a = \lambda_i^a \, \omega^i. \tag{3}$$

The functions $\lambda_i^a = \lambda_i^a(x, u)$ form the first order geometrical-differential object of f [11].

Submersion f defines on the manifold M the foliation Φ with leafs of codimension m, and X is the base of the foliation. A leaf of Φ is located by fixation of a point of X, that is, by the equations u^a =const. These equations are equivalent to $\vartheta^a = 0$, and from (3) the differential equations of the foliation Φ follow:

$$\lambda_i^a \, \omega^i = 0. \tag{4}$$

Pseudogroup of local diffeomorphisms acts on the manifold M, we denote it by $\mathcal{P}$. Let $\mathcal{Q}$ be the similar group, acting on X. If there are not any additional structures on manifolds M and X, then submersion f and foliation Φ must be considered up to transformations of the pseudogroup $\mathcal{P} \times \mathcal{Q}$.

The following statement is given without proof.

Theorem 1. *Let $f : M \to X$ be a smooth submersion, and the structure equations of the manifolds M and X are written as (1) and (2). Then the bases of the co-frame bundles of the first, second, etc. orders on manifolds M and X can be chosen in such a way that the equations*

$$\vartheta^a = \omega^a, \quad \vartheta_b^a = \omega_b^a, \quad \omega_{bc}^a = \vartheta_{bc}^a, \ldots$$

hold for any number of sub-indices.

If the above relations hold we will say that *the structures of manifolds M and X canonically agree with respect to submersion f*. In this case the structure equations (2) of manifold X are part of the structure equations (1) of the manifold M, which we will call *the canonical structure equations of submersion f or foliation* Φ.

3. Invariants of 2-dimensional foliation in three-dimensional Euclidean space

3.1 The equations of submersion $f : \mathbb{E}^3 \to \mathbb{R}$

Let $f : \mathbb{E}^3 \to \mathbb{R}$ be a smooth function and $\mathbb{E}^3$ be the three-dimensional Euclidean space. In $\mathbb{E}^3$, we will use orthonormal frames. Therefore, the previous arguments have to change a little, as now we put some additional conditions on the frame bundles.

Let p be an arbitrary point of $\mathbb{E}^3$ and simultaneously p be the radius vector of this point; e_i be the moving orthonormal frame, $i, j, k, ... = 1, 2, 3$. We put, as usual,

$$dp = \omega^i e_i, \quad de_i = \omega_i^j e_j. \tag{5}$$

The 1-forms ω^i and ω_i^j satisfy the structure equations of Euclidean space:

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_j^i = \omega_j^k \wedge \omega_k^i, \tag{6}$$

and the relations

$$\omega_j^i = -\omega_i^j,\, \omega_i^i = 0, \tag{7}$$

which are derived from orthonormality conditions. It is known [14] that the forms ω^i and ω_j^i are the invariant forms of the group D^3 of Euclidean motions.

In the case considered $m = 1$, so the equations (2) take the form

$$d\vartheta^1 = \vartheta^1 \wedge \vartheta_1^1,\, d\vartheta_1^1 = \vartheta^1 \wedge \vartheta_{11}^1,\, d\vartheta_{11}^1 = \vartheta_1^1 \wedge \vartheta_{11}^1 + \vartheta^1 \wedge \vartheta_{111}^1, \dots \tag{8}$$

If we change the given co-bases on $\mathbb{R}$ by the relations $\tilde{\vartheta}^1 = q\vartheta^1$ then the forms $\vartheta_1^1, \vartheta_{11}^1, \dots,$ are transformed as follows:

$$\tilde{\vartheta}_1^1 = \vartheta_1^1 - dq/q,\, \tilde{\vartheta}_{11}^1 = q^{-1}\vartheta_{11}^1,\, \tilde{\vartheta}_{111}^1 = q^{-2}\vartheta_{111}^1, \dots .$$

There are two reasons for adapting the family of orthonormal frames in $\mathbb{E}^3$ 1) to simplify the equations (4) of foliation Φ, and 2) to simplify the equations of submersion f. Usually, the leaf V (a two-dimensional surface) in $\mathbb{E}^3$ is given by the equation $\omega^3 = 0$. In this case vectors e_1 and e_2 of the moving frame tangent to V, and vector e_3 is orthogonal to V. In this case the admissible transformations have the form

$$\tilde{\omega}^3 = \omega^3,\, \tilde{\omega}^a = p_b^a \omega^b,$$

where $a, b = 1, 2$ and (p_b^a) is an orthogonal matrix. By choosing the family of frames, we get the equations of submersion in the form $\vartheta^1 = \lambda\omega^3$. By replacing $\vartheta^1 \to \lambda\vartheta^1$ we reduce these submersion equation to the form

$$\vartheta^1 = \omega^3. \tag{9}$$

On the other hand, an arbitrary diffeomorphism $\mathcal{Q} \to \mathcal{Q}$ should be written in the form $\tilde{\vartheta}^1 = q\vartheta^1$ or, in view of (9), in the form $\tilde{\omega}^3 = q\omega^3$. Comparing it with the above mentioned admissible transformations we find $q = 1$, so every difffeomorfism $\mathcal{Q} \to \mathcal{Q}$ can be written in the form

$$\tilde{\omega}^3 = \omega^3. \tag{10}$$

It follows that the pseudogroup of the gauge transformations $\vartheta^1 \to q\vartheta^1$ on $\mathbb{R}$ is trivial, that is, the calibration is fixed. We can see from these considerations that non-trivial gauge transformations arise only in the case when a non-orthonormal moving frame is used, that is, the normal vector field on V is not necessarily a unit one. In what follows we put $q = 1$ (the calibration is fixed) then the equations (9) and (10) are fulfilled.

3.2 The canonical frame of submersion $f : \mathbb{E}^3 \to \mathbb{R}$

Exterior differentiation of equation (9) by means of (6)–(8) in view of (9) leads to

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \omega^3 \wedge \vartheta_1^1.$$

Cartan's lemma implies:

$$\omega_1^3 = a_{11}\omega^1 + a_{12}\omega^2 + a_1\omega^3,\ \omega_2^3 = a_{21}\omega^1 + a_{22}\omega^2 + a_2\omega^3,\ a_{12} = a_{21}, \tag{11}$$

and

$$\vartheta_1^1 = -a_1\omega^1 - a_2\omega^2 + a_3\omega^3. \tag{12}$$

Note that by substituting this expression into the first equation of system (8) we obtain an expression for the form $d\omega^3$, which follows also from the first series of the equations (6) and the equations (11):

$$d\omega^3 = a_1\omega^1 \wedge \omega^3 + a_2\omega^2 \wedge \omega^3. \tag{13}$$

Exterior differentiation of equations (11) gives

$$\begin{aligned} \bigtriangledown a_{11} \wedge \omega^1 + \bigtriangledown a_{12} \wedge \omega^2 + \bigtriangledown a_1 \wedge \omega^3 = 0, \\ \bigtriangledown a_{12} \wedge \omega^1 + \bigtriangledown a_{22} \wedge \omega^2 + \bigtriangledown a_2 \wedge \omega^3 = 0. \end{aligned}$$

Here

$$\begin{aligned} \bigtriangledown a_{11} &= da_{11} - 2a_{12}\omega_1^2, \\ \bigtriangledown a_{12} &= da_{12} - a_{11}\omega_2^1 - a_{22}\omega_1^2, \\ \bigtriangledown a_{22} &= da_{22} - 2a_{12}\omega_2^1, \\ \bigtriangledown a_1 &= da_1 - a_2\omega_1^2 + a_{11}\omega_1^3 + a_{12}\omega_2^3 + (a_1)^2\omega^1 + a_1a_2\omega^2, \\ \bigtriangledown a_2 &= da_2 - a_1\omega_2^1 + a_{12}\omega_1^3 + a_{22}\omega_2^3 + a_1a_2\omega^1 + (a_2)^2\omega^2. \end{aligned} \tag{14}$$

Using Cartan's lemma from the above quadratic equations we obtain

$$\begin{aligned} \bigtriangledown a_{11} &= a_{111}\omega^1 + a_{112}\omega^2 + a_{113}\omega^3, \\ \bigtriangledown a_{12} &= a_{112}\omega^1 + a_{122}\omega^2 + a_{123}\omega^3, \\ \bigtriangledown a_{22} &= a_{122}\omega^1 + a_{222}\omega^2 + a_{223}\omega^3, \\ \bigtriangledown a_1 &= a_{113}\omega^1 + a_{123}\omega^2 + b_{11}\omega^3, \\ \bigtriangledown a_2 &= a_{123}\omega^1 + a_{223}\omega^2 + b_{22}\omega^3. \end{aligned} \tag{15}$$

By differential prolongation of the equation (12) we get

$$\begin{aligned} &da_3 + \vartheta_{11}^1 = a_{33}\omega^3 + \\ &(-b_{11} - a_1a_3 + 2a_1a_{11} + 2a_2a_{12})\omega^1 + (-b_{22} - a_2a_3 + 2a_1a_{12} + 2a_2a_{22})\omega^2, \end{aligned} \tag{16}$$

where a_{33} is a new function.

Fix a point p by setting $\omega^i = 0$. Then from (16) we get the equation

$$\delta a_3 + \pi_{11}^1 = 0,$$

where, as usual, $\pi^i_j = \omega^i_j (\mathrm{mod}\, \omega^i)$, and δ is the symbol of differentiation with respect to secondary parameters. This equality shows that by fixing the variable a_3 we can reduce the form π^1_{11} to zero. In other words, we can constrict the family of frames, using only those in which the form ϑ^1_{11} is the principal one. Suppose, for example, $a_3 = 0$, then from (16) we obtain the expression for the form ϑ^1_{11}:

$$\vartheta^1_{11} = a_{33}\omega^3 + (-b_{11} + 2a_1a_{11} + 2a_2a_{12})\omega^1 + (-b_{22} + 2a_1a_{12} + 2a_2a_{22})\omega^2. \tag{17}$$

As a result of such a canonization the equation (12) takes the form:

$$\vartheta^1_1 = -a_1\omega^1 - a_2\omega^2. \tag{18}$$

If we differentiate the equation (18) using the second equation (8), equation (16) (under the condition $a_3 = 0$), the last two equations (14) and structure equations (6), we obtain the identity.

Next, we reason similarly. We differentiate (17), next apply Cartan's lemma and put in the resulting equations $\omega^i = 0$. After calculations we obtain the equality

$$\delta a_{33} + \pi^1_{111} = 0.$$

By fixing $a_{33} = 0$ we can reduce the form π^1_{111} to zero. Then the equation (17) takes the form

$$\vartheta^1_{11} = (-b_{11} + 2a_1a_{11} + 2a_2a_{12})\omega^1 + (-b_{22} + 2a_1a_{12} + 2a_2a_{22})\omega^2. \tag{19}$$

Continuing the induction argument we obtain the following

Theorem 2. *Let $f : \mathbb{E}^3 \to \mathbb{R}$ be a submersion, and the space $\mathbb{E}^3$ is related to orthonormal frame, and structure equations of pseudo-group transformations on $\mathbb{R}$ are written in the form* (8). *Then the pseudo-group of gauge transformations is trivial, and the families of frames on $\mathbb{E}^3$ and on $\mathbb{R}$ can be chosen so that the form ω^3 is the principal one on $\mathbb{R}$ and there are the frames of the first, second, etc. orders such that the equations* (9), (12), (19) *and etc hold, and these equations are completely integrable on the manifold $\mathbb{E}^3 \times \mathbb{R}$.*

We will call the frame indicated in the Theorem 2 *the canonical frame $\mathcal{R}_\infty$ of the submersion $f : \mathbb{E}^3 \to \mathbb{R}$*. If only equation (9) is fulfilled, we will say that there is a canonization of the first order and will denote the corresponding frames by $\mathcal{R}_1$. If the equations (9)and (18) are fulfilled we will say that there is a canonization of the second order and will denote the corresponding frames by $\mathcal{R}_2$, etc.

Explain the geometric meaning of the canonization. We put $\omega^1 = \omega^2 = 0$, then from the first series of equations (5) we get $dp = \omega^3 e_3$ and from (13) we get $d\omega^3 = 0$, which implies $\omega^3 = ds$. Consequently, the point p describes an orthogonal trajectory (we denote it by ℓ) of the foliation Φ and s is the natural parameter on the ℓ. On the other hand, since the forms ϑ^1_1, $\vartheta^1_{11}, \ldots$ are zero by the condition $\omega^1 = \omega^2 = 0$, equations (8) are reduced to the single equation $d\vartheta^1 = 0$ or $d\omega^3 = 0$. So, a structure of one-dimensional Euclidean space with the canonical parameter s (Cartesian coordinate) arises on $\mathbb{R}$. *Thus the manifold $\mathbb{R}$ being the base of the foliation Φ is canonically embedded in $\mathbb{E}^3$ as a orthogonal trajectory of this foliation.*

3.3 Pseudogroup of transformations in $\mathbb{E}^3 \times \mathbb{R}$

As is already noted, we considered the submersion of f and the foliation Φ up to transformations of pseudo-group $\mathcal{P} \times \mathcal{Q}$. In our case, $\mathcal{P}$ is the group D^3 of Euclidean motions. In terms of invariant forms ω^i and ω^i_j the action of group D^3 can be written in the form

$$\tilde{\omega}^3 = \omega^3,\ \tilde{\omega}^a = p^a_b \omega^b,$$

where $a, b = 1, 2$ and (p^a_b) is an orthogonal matrix. The last equations can be simplified by choosing the canonical frame, whose vectors e_1 and e_2 are tangent to leaf V along the principal directions of V. Then the previous equations take the form

$$\tilde{\omega}^i = \omega^i. \tag{20}$$

By differentiating these equations with the help of structure equations (6) in view of (20) we obtain $\omega^j \wedge (\omega^i_j - \tilde{\omega}^i_j) = 0$. From that we get $\omega^i_j - \tilde{\omega}^i_j = c^i_{jk}\omega^k$, $c^i_{jk} = c^i_{kj}$. On the other hand, by (7) we get $c^i_{jk} = -c^j_{ik}$. We have: $c^i_{jk} = -c^j_{ik} = -c^j_{ki} = c^k_{ji} = c^k_{ij} = -c^i_{kj} = -c^i_{jk}$, so $c^i_{jk} = 0$ and we get the equations $\omega^i_j = \tilde{\omega}^i_j$. By integrating these equations together with the equations (18) we obtain, see [15], the action of the group D^3 on itself.

An arbitrary diffeomorphism $\mathbb{R} \to \mathbb{R}$ of $\mathcal{Q}$ has the form (10) or

$$\tilde{\vartheta}^1 = \vartheta^1. \tag{21}$$

Differential prolongation of this equation gives $\tilde{\vartheta}^1_1 = \vartheta^1_1 + k\omega^3$ or

$$\tilde{a}_1\tilde{\omega}^1 + \tilde{a}_2\tilde{\omega}^2 = a_1\omega^1 + a_2\omega^2 + k\omega^3.$$

Since the forms ω^1 and ω^2 allow only orthogonal transformation of the form $\tilde{\omega}^a = p^a_b\omega^b$ (see above), we have $k = 0$ and co-vector $\tilde{\mathbf{a}}(\tilde{a}_1, \tilde{a}_2)$ is obtained from covector $\mathbf{a}(a_1, a_2)$ by inverse orthogonal transformation. We write that in the form

$$\tilde{\mathbf{a}} = \Pi(\mathbf{a}). \tag{22}$$

This equation is equivalent to

$$\tilde{\vartheta}^1_1 = \vartheta^1_1. \tag{23}$$

In view of (19) differential prolongation of equation (23) gives

$$\tilde{\mathbf{B}}_{11} = \Pi(\mathbf{B}_{11}), \tag{24}$$

where $\mathbf{B}_{11}$ is a co-vector with coordinates

$$(-b_{11} + 2a_1a_{11} + 2a_2a_{12},\ -b_{22} + 2a_1a_{12} + 2a_2a_{22}),$$

and the co-vector $\tilde{\mathbf{B}}_{11}$ has the same coordinates with the wave.

On induction, we obtain the sequence of co-vectors $\mathbf{a}$, $\mathbf{B}_{11}$, $\mathbf{B}_{111}\ldots$, which are converted by the operator Π into the similar quantities, but calculated at a different point.

We proved

Theorem 3. *Let a submersion $f : \mathbb{E}^3 \to \mathbb{R}$ with trivial pseudo-group of gauge transformations be related to the canonical frame $\mathcal{R}_\infty$. Then the action of pseudo-group transformations on $\mathbb{R}$ is the direct product of transformation (10), which transform the leaf V into the leaf $\tilde{V}$, and the operator Π, that "transplant" invariant co-vectors from leaf V into the leaf $\tilde{V}$ along orthogonal trajectories.*

3.4 Invariants of the foliation under group D^3 action

Equations (14) and (15) imply

$$\delta a_{11} - 2a_{12}\pi_1^2 = 0,\ \delta a_{12} - a_{11}\pi_2^1 - a_{22}\pi_1^2 = 0,\ \delta a_{22} - 2a_{12}\pi_2^1 = 0, \tag{25}$$

where the symbols δ and π have the same meaning as above. From equations (25) we get the equations

$$\delta(a_{11}a_{22} - (a_{12})^2) = 0, \quad \delta(a_{11} + a_{22}) = 0,$$

which means that the functions $a_{11}a_{22} - (a_{12})^2$ and $a_{11} + a_{22}$ are invariants with respect to admissible transformations of the frame (coframe).

Equations (25) can be written as

$$\delta a_{ab} + a_{cb}\pi_a^c + a_{ac}\pi_b^c = 0,\ \delta a_a + a_b\pi_a^b = 0.$$

The equations show that the quantities a_{ab} and a_a form tensors with respect to admissible transformations. Let us find geometric meaning of the tensors.

Completely integrable equation

$$\omega^3 = 0 \tag{26}$$

allocates a leaf of foliation Φ that is a two-dimensional surface V. On V, equations (11) take the form

$$\omega_1^3 = a_{11}\omega^1 + a_{12}\omega^2,\ \omega_2^3 = a_{21}\omega^1 + a_{22}\omega^2. \tag{27}$$

Using (16) and (27) we find from (5)

$$dp = \omega^1 e_1 + \omega^2 e_2, \quad d^2p = (...)e_1 + (...)e_2 + a_{11}(\omega^1)^2 + 2a_{12}\omega^1\omega^2 + a_{22}(\omega^2)^2.$$

As we can see, the tensor a_{uv} is the asymptotic tensor of the surface V, and the asymptotic form of V is $\varphi_2 = a_{11}(\omega^1)^2 + 2a_{12}\omega^1\omega^2 + a_{22}(\omega^2)^2$. The first quadratic form of V is $(dp)^2 = (\omega^1)^2 + (\omega^2)^2$. Hence it is easy to find Gauss curvature and mean curvature of V:

$$K = a_{11}a_{22} - (a_{12})^2, \quad 2H = a_{11} + a_{22}.$$

Let us consider the domain of non-umbilical points on V and restrict the family of adapted frames by putting $a_{12} = 0$. Then we get the so-called canonical frame of the surface V. In this case from (25) we find $\delta a_{11} = \delta a_{22} = 0$, that is, the values

a_{11} and a_{22} became the invariants. It is well known that these are the principal curvatures of the surface V. Because the surface V is not umbilical we have $k_1 \neq k_2$ and the form ω_1^2 can be found from the second equation (15). In this case, the coordinate directions on V defined by vectors e_1 and e_2 are the principal directions of V.

To explain the geometric meaning of the invariant co-vector a_u, consider the foliation $\Phi^\perp$ of orthogonal trajectories of the foliation Φ. It is given by the equations

$$\omega^1 = \omega^2 = 0. \tag{28}$$

Let, as above, ℓ be an arbitrary curve from $\Phi^\perp$, $p \in \ell$. In view of (28) we get $dp = \omega^3 e_3, \omega^3 = ds$. Taking into account (7) and (11) we have

$$\frac{dp}{ds} = e^3, \quad \frac{d^2p}{ds^2} = \frac{\omega_3^1 e_1 \omega_3^2 e_2}{\omega^3} = -a_1 e_1 - a_2 e_2.$$

As we can see, the vector $-a_1e_1 - a_2e_2$ is the so-called curvature vector of ℓ. Two invariants are associated with it: the curvature $k = |-a_1e_1 - a_2e_2| = ((a_1)^2 + (a_2)^2)^{\frac{1}{2}}$ of the curve ℓ and the angle between curvature vector and one of the principal directories surface V. In the canonical frame the values a_1 and a_2 will be the invariants, namely, the projections of curvature vector on the principal directions of V.

By similar reasoning we can find the geometrical meaning of differential invariants of arbitrary order.

3.5 Algebra of invariants of foliation $f : \mathbb{E}^3 \to \mathbb{R}$

Consider in $\mathbb{E}^3$ the canonical frame, then the equality $a_{12} = 0$ is fulfilled. In this case, all functions at the structure equations (11), (15) and the subsequent equations received by standard differential prolongation are the invariants with respect to group $D^3 \times \mathcal{Q}$ action, where D^3 is the group of Euclidean motions of $\mathbb{E}^3$ and $\mathcal{Q}$ is the pseudo-group of local diffeomorphisms of $\mathbb{R}$. In the differential neighbourhood of the second order there are 4 independent invariants: a_{11}, a_{22}, a_1 and a_2.

To find the differential invariants in the differential neighbourhood of the third order, rewrite equation (15) using (14) and the condition $a_{12} = 0$. After some calculations transformations we get

$$\begin{aligned}
da_{11} &= a_{111}\omega^1 + a_{112}\omega^2 + a_{113}\omega^3, \\
da_{22} &= a_{122}\omega^1 + a_{222}\omega^2 + a_{223}\omega^3, \\
da_1 &= (aa_2a_{112} - (a_1)^2 - (a_{11})^2 + a_{113})\omega^1 + \\
&\quad (aa_2a_{122} - a_1a_2 + a_{123})\omega^2 + \tilde{b}_{11}\omega^3, \\
da_2 &= (-aa_1a_{112} - a_1a_2 + a_{123})\omega^1 + \\
&\quad (-aa_1a_{122} - (a_2)^2 - (a_{22})^2 + a_{223})\omega^2 + \tilde{b}_{22}\omega^3,
\end{aligned} \tag{29}$$

where

$$a = (a_{11} - a_{22})^{-1},\ \tilde{b}_{11} = aa_2a_{123} - a_1a_{11} + b_{11},\ \tilde{b}_{22} = -aa_1a_{123} - a_2a_{22} + b_{22}.$$

Coefficients of the basic forms in the right sides of equations (29) are invariants of the differential neighbourhood of the third order. As we can see, they are derivatives of the invariants with respect to invariant directions, that is, by differentiation along the principal directions of the surface V and along the curve ℓ.

As we can see from equations (15), in the differential neighbourhood of the third order only 12 invariants can be obtained, 9 of them $(a_{111}, a_{112}, a_{113}, a_{122}, a_{222}, a_{223}, a_{123}, \tilde{b}_{11}\tilde{b}_{22})$ being independent, and the remaining 3 are expressed in terms of the first 9 invariants. Coordinates of the co-vectors $\mathbf{a}$, $\mathbf{B}_{11}$, $\mathbf{B}_{111}\ldots$, (being invariants) are also expressed in terms of the above mentioned 9 invariants.

In the same way, other invariants can be found, that is, *by differentiation of the invariants of the previous differential neighbourhood along the principal directions.* In this case we say that the invariants algebra is generated by the invariants of a_{11}, a_{22}, a_1 and a_2.

A role of the invariants a_{11}, a_{22}, a_1 and a_2 can be seen from the following statement.

Equivalence Theorem. *Let there be given the foliation $f : \mathbb{E}^3 \to \mathbb{R}$, with the structure equations* (6), (7), (11), (15)*, etc., and the foliation $\tilde{f} : \tilde{\mathbb{E}}^3 \to \tilde{\mathbb{R}}$, defined by structure equations of the same form but in which the forms ω^i_j and the functions a_{11}, a_{22}, a_1, a_2, etc. are equipped by tilde. Suppose the corresponding invariants of the differential neighbourhood of the second order of these foliations coincide. Then foliations f and $\tilde{f}$ are equivalent, i.e., there is a Euclidean motion which the foliation f transform into foliation $\tilde{f}$.*

□ For simplicity we assume that the structure equations of both foliations are written in the canonical frame $(a_{12} = \tilde{a}_{12} = 0)$. Then, under the hypothesis of the theorem, we can write the equalities

$$a_{11} = \tilde{a}_{11}, a_{22} = \tilde{a}_{22}, a_1 = \tilde{a}_1, a_2 = \tilde{a}_2. \tag{30}$$

Let us consider the equations

$$\tilde{\omega}^i = \omega^i, \tag{31}$$

on the direct product $\tilde{\mathbb{E}}^3 \times \tilde{\mathbb{E}}^3$. By (30) and (31) from (11) we obtain $\omega^3_1 = \tilde{\omega}^3_1$, $\omega^3_2 = \tilde{\omega}^3_2$, and from (28) — $a_{111} = \tilde{a}_{111}$, $a_{112} = \tilde{a}_{112}$, etc. In their turn, these equations imply the equality $\omega^2_1 = \tilde{\omega}^2_1$. Using the obtained equality we get the relation $d(\omega^i - \tilde{\omega}^i) = 0$. Therefore, by Frobenius' theorem, the system (31) is completely integrable. Its integral manifold is the graph of the mapping $\varphi : \tilde{\mathbb{E}}^3 \to \tilde{\mathbb{E}}^3$ which is uniquely determined by given initial conditions. But equations (31) coincide with equations (20), which (see Section 3.3) determine the group of Euclidean motions. As can we see from the equations (31), φ maps the leaves of foliation Φ onto the leaves of foliation $\widetilde{\Phi}$. ■

4. Conclusion

Similar results can be obtained if the pseudogroup of gauge transformations is not trivial.

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